1 Simultaneous Diagonalization

Let $V$ be a finite dimensional vector space over a field $F$. Suppose that $\mathcal{A}$ is a set whose elements are operators $T_i : V \to V$ such that for any $T_i, T_j \in \mathcal{A}$, we have $T_i T_j = T_j T_i$.

(In other words, any two operators in $\mathcal{A}$ commute). Furthermore, suppose that every $T_i \in \mathcal{A}$ is diagonalizable.

The following theorem, despite its tricky proof and not so immediate-to-appreciate statement, is extremely useful. There are numerous applications, maybe some of the most remarkable ones being in quantum mechanics.

**Theorem 1.** Under the assumptions above, there exists a basis $\mathcal{B}$ with respect to which all $T_i \in \mathcal{A}$ are diagonal. (We say that the operators in $\mathcal{A}$ are simultaneously diagonalizable.)

**Proof:** The proof is by induction on $\dim(V) = n$. For $n = 1$, everything is clear: With respect to any basis for $V$ (which just has 1 element), all $T_i$’s will act as scalar multiplications, so they will all be diagonal. Also, if $\mathcal{A}$ contains only scalar multiples $\lambda I$ of the identity operator (for various values of $\lambda \in F$), then the statement clearly holds again.

Suppose now that $n \geq 2$ and the statement holds for any vector space with dimension less than $n$. Say $T \in \mathcal{A}$ is an operator which is not of the form $\lambda I$ for any $\lambda$. Since $T$ is diagonalizable, the minimal polynomial of $T$ is a product of distinct linear factors:

$$\delta_T(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_k)$$

where $k \geq 2$ and $\lambda_1, \ldots, \lambda_k$ are distinct. $V$ is the direct sum of eigenspaces of $T$, so let us write

$$V = W_{\lambda_1} \oplus W_{\lambda_2} \oplus \cdots \oplus W_{\lambda_k}$$

where $W_{\lambda_i}$ is the eigenspace associated to $\lambda_i$. Notice that $\dim(W_{\lambda_i}) < \dim(V)$ for each $i$ since $k \geq 2$. Suppose now that $U \in \mathcal{A}$ is any other operator.

Claim: Each $W_{\lambda_i}$ is an invariant subspace for $U$.

Proof of claim: Suppose that $w \in W_{\lambda_i}$. Then

$$T(Uw) = U(Tw) = U(\lambda_i w) = \lambda_i Uw.$$ 

But this implies, by definition of the eigenspace $W_{\lambda_i}$, that $Uw \in W_{\lambda_i}$. Hence the claim is proved.

Next, let us restrict each operator $U \in \mathcal{A}$ to the eigenspace $W_{\lambda_i}$. The new set $\mathcal{A}' = \{U|_{W_{\lambda_i}} \}_{U \in \mathcal{A}}$ is a new set of commuting operators on a lower dimensional vector space $W_{\lambda_i}$, and they are diagonalizable. Hence, by the inductive assumption, there exists a basis $\mathcal{B}_{\lambda_i}$ for $W_{\lambda_i}$ simultaneously diagonalizing all operators in $\mathcal{A}'$. This is true for each $W_{\lambda_i}$. It is easy to check that if we take

$$\mathcal{B} = \bigcup \mathcal{B}_{\lambda_i}$$

then all elements of $\mathcal{A}$ are diagonalized with respect to $\mathcal{B}$. This finishes the proof of the theorem.
Exercise: Let $A = \{T_\theta\}_{\theta \in [0, 2\pi]}$, where $T_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ denotes rotation about the $z$-axis by $\theta$ radians. Show that any two elements of $A$ commute. Show that in the standard basis $T_\theta$ is represented by the matrix

$$A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ 

Show that except finitely many values of $\theta$ the elements of $A$ are not diagonalizable. However, show that the operators $\hat{T}_\theta : \mathbb{C}^3 \to \mathbb{C}^3$ represented by the same matrices $A_\theta$ above with respect to the standard basis of $\mathbb{C}^3$ are diagonalizable. Find a basis $B$ for $\mathbb{C}^3$ with respect to which all $\hat{T}_\theta$ are simultaneously diagonalizable.
1 Motivation: The dot product on $V = \mathbb{R}^3$

Let $V = \mathbb{R}^3$ be the vector space over the field of real numbers with the usual operations. Recall that a very useful notion about vector geometry in $\mathbb{R}^3$ is the dot product. The precise definition is as follows: If $v = (x_1, y_1, z_1)$ and $w = (x_2, y_2, z_2)$ are two vectors in $\mathbb{R}^3$, then their dot product is the real number given by the formula

$$v \cdot w = x_1x_2 + y_1y_2 + z_1z_2.$$ 

The dot product can be used to compute familiar geometric quantities. For example, if $v = (x, y, z)$ then its (Euclidean) length $||v||$ satisfies

$$||v||^2 = v \cdot v = x^2 + y^2 + z^2.$$ 

therefore this length can be computed just by using the dot product. Another important quantity is the angle between two vectors. If the angle between two nonzero vectors $v = (x_1, y_1, z_1)$ and $w = (x_2, y_2, z_2)$ is $\theta$, then the following formula is well-known:

$$v \cdot w = ||v|| ||w|| \cos \theta.$$ 

(This formula can be proven in several ways, the easiest is probably by using the cosine law for triangles in Euclidean geometry.) Since the lengths in this formula can be expressed also in terms of the dot product, this implies that any such angle can be calculated just by using the dot product:

$$\cos \theta = \frac{v \cdot w}{\sqrt{v \cdot v} \sqrt{w \cdot w}}.$$ 

Notice that it is essential for the quantities $v \cdot v$ and $w \cdot w$ to be strictly positive for this formula to make sense. This can be checked by direct calculation (remember that $v$ and $w$ are nonzero vectors).

An important special case is when $\theta$ is a right angle, namely when $v$ and $w$ are orthogonal. This happens if and only if $\cos \theta = 0$. Hence, by the formula above, $v$ and $w$ are orthogonal if and only if $v \cdot w = 0$.

Let us now consider orthogonal projection: Suppose that $v$ and $w$ are nonzero vectors in $\mathbb{R}^3$. The orthogonal projection of $v$ along $w$ is a vector, which we will denote by $proj_w(v)$, characterized by the following two properties:
2 Inner Products on a Real Vector Space

- \( \text{proj}_w(v) \) is a scalar multiple of \( w \)
- The difference vector \( v - \text{proj}_w(v) \) is orthogonal to \( w \).

We can easily compute \( \text{proj}_w(v) \) explicitly based on these two properties: If \( \text{proj}_w(v) \) is a scalar multiple of \( w \), then \( \text{proj}_w(v) = \lambda w \) for some \( \lambda \in \mathbb{R} \). The second condition says that \( v - \lambda w \) is orthogonal to \( w \). Expressing this condition in terms of the dot product, we get

\[
(v - \lambda w) \cdot w = 0
\]
\[
(v \cdot w) - \lambda (w \cdot w) = 0
\]
\[
\lambda = \frac{v \cdot w}{w \cdot w}
\]

Therefore we get the following formula for \( \text{proj}_w(v) \), again purely in terms of the dot product:

\[
\text{proj}_w(v) = \frac{v \cdot w}{w \cdot w}w.
\]

2 Inner Products on a Real Vector Space

Now, we would like to give an abstract definition which would generalize the concept of a dot product to any vector space over the field of real numbers, which is as general as possible and at the same time still allows us to define and compute the geometric quantities mentioned in the previous section (length, angle, orthogonality, orthogonal projection). The key question is “what are the critically important features of the dot product that made things work?” The definition agreed upon turns out to be the following one:

**Definition 1.** Let \( V \) be a vector space over the field \( \mathbb{R} \). An **inner product** on \( V \) is a function \( V \times V \to \mathbb{R} \) taking a pair of vectors \( (v, w) \) to a real number which will be denoted by \( \langle v, w \rangle \), satisfying the following properties

1. \( \langle c_1 v_1 + c_2 v_2, w \rangle = c_1 \langle v_1, w \rangle + c_2 \langle v_2, w \rangle \) for any \( v_1, v_2, w \in V \) and \( c_1, c_2 \in \mathbb{R} \).
2. \( \langle v, w \rangle = \langle w, v \rangle \) for any \( v, w \in V \).
3. \( \langle v, v \rangle \geq 0 \) for any \( v \in V \). Furthermore, equality happens if and only if \( v \) is the zero vector.

We will give some key examples now. I will leave it to you as exercise to check the details. We will develop the geometric notions (length, angle, orthogonality, orthogonal projection etc.) and do much more in subsequent lectures.

**Exercise 1:** Let \( V = \mathbb{R}^n \). The **standard inner product** on \( V \) is defined as follows: If \( v = (x_1, x_2, \ldots, x_n) \) and \( w = (y_1, y_2, \ldots, y_n) \), then

\[
\langle v, w \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n.
\]

Show that the formula above indeed defines an inner product on \( \mathbb{R}^n \).

**Exercise 2:** Let \( V = M_{n \times n}(\mathbb{R}) \) be the vector space of \( n \times n \) matrices over real numbers, with respect to the usual operations. Show that the formula

\[
\langle A, B \rangle = Tr(B^T A)
\]
defines an inner product on $V$.

**Exercise 3:** Let $V = C([a, b])$ denote the vector space of continuous functions on the interval $[a, b]$, where $a < b$. For any $f, g \in V$, define

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$ 

Show that this formula defines an inner product on $V$. (Caution: There is a tricky part in this exercise. As a hint, let me say that the claim would not be true if we didn’t assume the continuity of the functions. Where is continuity really used?)
1 Inner Product Spaces

Definition 1. Let \( V \) be a vector space over the field of real numbers, and let \( \langle \cdot, \cdot \rangle \) be an inner product on \( V \). Then the pair \( (V, \langle \cdot, \cdot \rangle) \) is called an inner product space.

1.1 Norm and Orthogonality

Definition 2. Let \( (V, \langle \cdot, \cdot \rangle) \) be an inner product space and \( v \in V \). Then the length (or norm) of \( v \) is
\[
||v|| = \sqrt{\langle v, v \rangle}.
\]
A vector of length 1 is called a unit vector.

Note that \( \langle v, v \rangle \geq 0 \) by one of the defining properties of an inner product, so that this definition makes sense and always produces a nonnegative number. Furthermore, again by the same property, the length of a vector is zero if and only if the vector is the zero vector.

Definition 3. Let \( (V, \langle \cdot, \cdot \rangle) \) be an inner product space. Two vectors \( v, w \in V \) are said to be orthogonal if \( \langle v, w \rangle = 0 \). If \( S \) is any nonempty subset of \( V \) then the orthogonal complement of \( S \) is the set of all vectors in \( V \) that are orthogonal to all vectors in \( S \), and this set is denoted by \( S^\perp \).

Proposition 1. Let \( (V, \langle \cdot, \cdot \rangle) \) be an inner product space and \( S \subset V \) any nonempty subset. Then the orthogonal complement \( S^\perp \) of \( S \) is a subspace of \( V \).

Proof: The zero vector is orthogonal to any vector. Indeed, if \( 0 \) denotes the zero vector, then
\[
\langle 0, v \rangle = \langle 0 + 0, v \rangle = \langle 0, v \rangle + \langle 0, v \rangle
\]
therefore \( \langle 0, v \rangle = 0 \). Therefore, \( 0 \in S^\perp \). Suppose now that \( w_1, w_2 \in S^\perp \). Then for any \( v \in S \), we have \( \langle w_1, v \rangle = 0 \) and \( \langle w_2, v \rangle = 0 \). Therefore,
\[
\langle w_1 + w_2, v \rangle = \langle w_1, v \rangle + \langle w_2, v \rangle = 0.
\]
So, \( w_1 + w_2 \in S^\perp \). Likewise, if \( w \in S^\perp \) and \( c \in \mathbb{R} \), then for any \( v \in S \)
\[
\langle cw, v \rangle = c \langle w, v \rangle = 0
\]
hence \( cw \in S^\perp \). Therefore, \( S^\perp \) is a subspace of \( V \). \( \Box \)

Example: Let \( V = C([-\pi, \pi]) \) be the set of real valued continuous functions on the interval \( [-\pi, \pi] \), viewed as a vector space over \( \mathbb{R} \). Equip \( V \) with the inner product
\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.
\]
Let us find the norm of \( f(x) = \sin x \) with respect to this inner product.

\[
||f||^2 = \langle f, f \rangle = \int_{-\pi}^{\pi} \sin^2(x) dx = \int_{-\pi}^{\pi} \frac{1}{2} - \frac{\cos(2x)}{2} dx = \frac{x}{2} - \frac{\sin(2x)}{4} \bigg|_{-\pi}^{\pi} = \pi.
\]

We deduce that \( ||f|| = \sqrt{\pi} \). We also claim that \( f(x) = \sin x \) and \( g(x) = \cos(x) \) are orthogonal. Indeed,

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} \sin x \cos x dx = \frac{\sin(2x)}{2} \bigg|_{-\pi}^{\pi} = 0.
\]

As an exercise, show that \( \sin(3x) \) and \( \sin(5x) \) are orthogonal vectors (can you generalize this example?).

### 1.2 Orthogonal Projection

**Proposition 2.** Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space, \( v, w \in V \) and \( w \) is not the zero vector. Then there is a unique vector \( \text{proj}_w v \in V \) such that

(i) \( \text{proj}_w v \) is a scalar multiple of \( w \),

(ii) \( v - \text{proj}_w v \) is orthogonal to \( w \).

Furthermore, \( \text{proj}_w v \) is given by the formula

\[
\text{proj}_w v = \frac{\langle v, w \rangle}{||w||^2} w.
\]

**Proof:** Existence: Let us show that the formula given above really satisfies (i) and (ii). Item (i) is clear since the formula for \( \text{proj}_w v \) above gives us a multiple of \( w \). To prove that (ii) is satisfied, look at

\[
\langle v - \text{proj}_w v, w \rangle = \langle v - \frac{\langle v, w \rangle}{||w||^2} w, w \rangle = \langle v, w \rangle - \frac{\langle v, w \rangle}{||w||^2} \langle w, w \rangle = \langle v, w \rangle - \langle v, w \rangle = 0.
\]

Uniqueness: Any vector satisfying (i) must be of the form \( \lambda w \) for some \( \lambda \in \mathbb{R} \). Now \( \lambda w \) satisfies (ii) if and only if \( \langle v - \lambda w, w \rangle = 0 \). But this happens if and only if \( \langle v, w \rangle = \lambda \langle w, w \rangle \), namely \( \lambda = \frac{\langle v, w \rangle}{||w||^2} \). \( \square \)
**Definition 4.** The vector $\text{proj}_w(v)$ in the proposition above is called the **orthogonal projection** of the vector $v$ along the vector $w$.

### 1.3 Cauchy-Schwarz inequality

We will now prove a fundamental inequality valid in any inner product space, which will allow us to show that geometric entities such as distance and angle behave as we would expect them to.

**Theorem 1.** (*Cauchy-Schwarz inequality*) Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space and $v,w \in V$. Then

$$|\langle v, w \rangle| \leq ||v|| ||w||.$$

Furthermore, equality occurs if and only if $v$ and $w$ are linearly dependent.

**Proof:** If $v$ or $w$ is the zero vector, then the statement clearly holds. Therefore, suppose that $v$ and $w$ are nonzero vectors. Recall that $\text{proj}_w v = \frac{\langle v, w \rangle}{||w||^2} w$. Let $\lambda = \frac{\langle v, w \rangle}{||w||^2}$. Then, using the fact that $\langle v - \lambda w, w \rangle = 0$, we have

$$0 \leq ||v - \lambda w||^2 = \langle v - \lambda w, v - \lambda w \rangle = \langle v - \lambda w, v \rangle = ||v||^2 - \lambda \langle w, v \rangle = ||v||^2 - \frac{(\langle v, w \rangle)^2}{||w||^2}.$$

This immediately gives us the desired inequality. Furthermore, in order to get equality, we must have $||v - \lambda w|| = 0$, which happens if and only if the two vectors $v$ and $w$ are linearly dependent.

**Corollary 1.** (*Triangle inequality*) Let $(V,\langle\cdot,\cdot\rangle)$ be an inner product space. Then for any $v,w \in V$, we have

$$||v + w|| \leq ||v|| + ||w||,$$

with equality if and only if $v = \lambda w$ for a nonnegative real number $\lambda$, or $w = \lambda v$ for $a$ or a nonnegative real number $\lambda$.

**Proof:**

$$||v + w||^2 = \langle v + w, v + w \rangle = ||v||^2 + 2\langle v,w \rangle + ||w||^2 \leq ||v||^2 + 2||v|| ||w|| + ||w||^2 = (||v|| + ||w||)^2.$$

This immediately gives us the triangle inequality. Above, Cauchy-Schwarz inequality was used exactly at the step where we have an inequality. In order to get equality, we must have $\langle v, w \rangle = ||v|| ||w||$, which happens if and only if one of $v$ or $w$ is a multiple of the other by a positive real number.

**Exercise:** Prove the "Pythagorean theorem", namely if $v,w \in V$ are orthogonal, then $||v + w||^2 = ||v||^2 + ||w||^2$. 

1.4 Angle Between Two Vectors

Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space and \(v, w \in V\). We would like to define the angle between the vectors \(v\) and \(w\). Motivated by the formula for dot product in \(\mathbb{R}^3\), we define the angle between \(v\) and \(w\) to be \(\theta\) (in radians) such that

\[
\cos \theta = \frac{\langle v, w \rangle}{||v||||w||}.
\]

The point is that, the Cauchy-Schwarz inequality guarantees us that the right hand side is in the interval \([-1, 1]\), therefore it is equal to \(\cos \theta\) for some \(\theta\).

**Exercise:** Let \(V = C([-\pi, \pi])\) be the set of real valued continuous functions on the interval \([-\pi, \pi]\), viewed as a vector space over \(\mathbb{R}\), equipped with the inner product

\[
\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.
\]

Find the angle between the vectors \(f(x) = 1\) (constant function) and \(g(x) = \sin^2(x)\). (Use a calculator to find the inverse cosine of some quantity if necessary.)
1 Hermitian Inner Products

There is a natural, and very useful, generalization of the concept of an inner product to a vector space over the field of complex numbers:

**Definition 1.** Let $F = \mathbb{C}$ and $V$ a vector space over $F$. A map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called a **Hermitian inner product** if it satisfies the following properties:

- $\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$ for all $v_1, v_2, w \in V$,
- $\langle cv, w \rangle = c \langle v, w \rangle$ for all $v, w \in V$ and $c \in \mathbb{C}$,
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$ where $\overline{a}$ denotes the complex conjugate of $a$,
- $\langle v, v \rangle$ is real and nonnegative for all $v \in V$. Furthermore, it is equal to zero if and only if $v$ is the zero vector.

Notice that, as opposed to inner products on real vector spaces, the two entries of a Hermitian inner product are not entirely symmetric: The relations $\langle v, w \rangle = \overline{\langle w, v \rangle}$ and $\langle cv, w \rangle = c \langle v, w \rangle$ imply $\langle v, cw \rangle = \overline{c} \langle v, w \rangle$.

On the other hand, it can be easily seen that $\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$.

The fact that $\langle v, v \rangle$ is a real number can actually be deduced from the previous rules: Putting $v = w$ in the third rule, we get $\langle v, v \rangle = \overline{\langle v, v \rangle}$. But any number which is equal to its complex conjugate must be a real number. On the other hand, the nonnegativity and the last sentence in the fourth rule must be postulated and does not follow from the previous rules.

**Example 1:** Let $V = \mathbb{C}^n$. Define

$$\langle (a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \rangle = a_1 \overline{b_1} + a_2 \overline{b_2} + \ldots + a_n \overline{b_n}.$$ 

Then check as exercise that $\langle \cdot, \cdot \rangle$ is a Hermitian inner product on $V$.

**Example 2:** Say $a < b$ are real numbers. Let $V$ denote the vector space of complex valued continuous functions on the interval $[a, b]$. Define

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx.$$ 

(If you haven’t yet seen the definition of the integral of a complex valued function on the interval $[a, b]$, it is just the quantity obtained by integrating the real and imaginary parts separately as real integrals and combining them.) Just as a concrete example, say $a = 0, b = 2, f(x) = x - ix$
and \( g(x) = 1 + ix \). Then
\[
\langle f, g \rangle = \int_0^2 f(x)\overline{g(x)}dx
\]
\[
= \int_0^2 (x - ix)(1 - ix)dx
\]
\[
= \int_0^2 (x - x^2) - i(x + x^2)dx
\]
\[
= \left( \frac{x^2}{2} - \frac{x^3}{3} \right) - i \left( \frac{x^2}{2} + \frac{x^3}{3} \right) \bigg|_0^2
\]
\[
= -\frac{2}{3} - \frac{14}{3}.
\]
Again, as an exercise, check that \( \langle \cdot, \cdot \rangle \) satisfies all conditions for being a Hermitian inner product.

**Definition 2.** Let \( V \) be a complex vector space and \( \langle \cdot, \cdot \rangle \) a Hermitian inner product on \( V \). Then the length (or norm) of a vector \( v \in V \) is defined to be
\[
||v|| = \sqrt{\langle v, v \rangle}.
\]
Since \( \langle v, v \rangle \) is a nonnegative real number as we observed above, it always has squareroot, which is itself nonnegative real. Therefore, the length of a vector is always a nonnegative real number. Furthermore, the length of a vector is zero if and only if the vector is the zero vector.

**Definition 3.** Let \( V \) be a complex vector space and \( \langle \cdot, \cdot \rangle \) a Hermitian inner product on \( V \). Say \( v, w \in V \). We say that \( v \) and \( w \) are orthogonal with respect to the Hermitian inner product if \( \langle v, w \rangle = 0 \).

**Example:** Let \( V \) be the set of complex valued continuous functions on the interval \([0, 2\pi]\) and recall the Hermitian inner product
\[
\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx
\]
as defined above. Recall Euler’s formula:
\[
e^{ix} = \cos(x) + i\sin(x).
\]
Then, \( e^{ix} \) is clearly an element of \( V \). Also notice that \( \overline{e^{ix}} = e^{-ix} \).

We claim that if \( m \neq n \) are two integers, then \( f(x) = e^{inx} \) and \( g(x) = e^{inx} \) are orthogonal.
Indeed,

$$\langle e^{inx}, e^{imx} \rangle = \int_0^{2\pi} e^{inx} e^{imx} \, dx$$

$$= \int_0^{2\pi} e^{inx} e^{-imx} \, dx$$

$$= \int_0^{2\pi} e^{i(n-m)x} \, dx$$

$$= \frac{1}{n-m} \sin((n-m)x) - \frac{i}{n-m} \cos((n-m)x) \bigg|_0^{2\pi}$$

$$= 0$$

Therefore, we found infinitely many vectors in $V$ any two of which are orthogonal to each other.

**Exercise:** In the example above, what was the significance of $n, m$ being integers (can’t they be any real numbers)? Where exactly did we use the assumption that $m \neq n$? What is the norm of the vector $e^{inx}$?

**Exercise:** Generalize the concepts of angle and orthogonal projection to a complex vector space equipped with a Hermitian inner product. Also, find correct generalizations the Cauchy-Schwarz inequality and the triangle inequality in this setting.
Orthogonal and Orthonormal Sets

Throughout the text, we assume that $V$ is either a real vector space with inner product $\langle \cdot, \cdot \rangle$, or a complex vector space with a Hermitian inner product $\langle \cdot, \cdot \rangle$.

**Definition 1.**
- A subset $S$ of $V$ is called an **orthogonal set** if any two elements of $S$ are orthogonal, namely if $\langle v, w \rangle = 0$ for any $v, w \in S$.
- A subset $S$ of $V$ is called an **orthonormal set** if it is an orthogonal set and every vector in $S$ has unit length, namely if $\langle v, w \rangle = 0$ for any $v, w \in S$ and $||v|| = 1$ for all $v \in S$.
- A subset $S$ of $V$ is called an **orthogonal basis** of $V$ if it is an orthogonal set and a basis of $V$.
- A subset $S$ of $V$ is called an **orthonormal basis** of $V$ if it is an orthonormal set and a basis of $V$.

**Example:** Let $V = \mathbb{R}^n$ together with its standard inner product. Then the standard basis vectors give us an orthonormal basis.

**Example:** Let $V$ be the set of complex valued continuous functions on $[0, 2\pi]$ which appeared in the last lecture, together with the Hermitian inner product

$$\langle f(x), g(x) \rangle = \int_0^{2\pi} f(x)\overline{g(x)}dx.$$  

We saw in that example that $e^{inx}$ is orthogonal to $e^{imx}$ if $n, m$ are integers and $n \neq m$. Therefore,

$$S = \{e^{inx}\}_{n \in \mathbb{Z}}$$

is an orthogonal set.

It is easy to obtain an orthonormal set from a given orthogonal set that doesn’t contain the zero vector: Observe that if $v$ is a nonzero vector, then $\frac{v}{||v||}$ is a unit vector. Indeed,

$$\langle \frac{v}{||v||}, \frac{v}{||v||} \rangle = \frac{1}{||v||^2} \langle v, v \rangle$$

$$= \frac{1}{||v||^2} ||v||^2$$

$$= 1.$$  

Therefore, if $S$ is an orthogonal set not containing the zero vector, then

$$\tilde{S} = \left\{ \frac{v}{||v||} \bigg| v \in S \right\}$$

is an orthonormal set. It is easy to see that if $S$ is an orthogonal basis, then $\tilde{S}$ is an orthonormal basis.

**Theorem 1.** Let $S$ be a subset of $V$ which is an orthogonal set and doesn’t contain the zero vector. Then, $S$ is linearly independent.
Proof: Suppose, to the contrary, that \( S \) is linearly dependent. Then there exists a finite subset of \( S \) which is linearly dependent. Choose a nontrivial dependence relation
\[
c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0
\]
where \( c_1, c_2, \ldots, c_n \) are real (or complex) numbers, not all zero, and \( v_i \in S \) for all \( i \). Evaluate the inner product of both sides of this equation by \( v_i \).
\[
\langle c_1v_1 + c_2v_2 + \ldots + c_nv_n, v_i \rangle = \langle 0, v_i \rangle
c_1\langle v_1, v_i \rangle + \ldots + c_i\langle v_{i-1}, v_i \rangle + c_i\langle v_i, v_i \rangle + c_{i+1}\langle v_{i+1}, v_i \rangle + \ldots + c_n\langle v_n, v_i \rangle = 0
\]
\[
c_i\|v_i\|^2 = 0
c_i = 0.
\]
(In this derivation we used three facts: First, the inner product (or Hermitian inner product) is linear with respect to the first entry. Second, \( \langle v_j, v_i \rangle = 0 \) for any \( i \neq j \) since \( S \) is an orthogonal set. Third, \( \|v_i\|^2 \neq 0 \) since we are in an inner product space (or Hermitian inner product space) and \( v_i \) is not the zero vector.) Now, since \( c_i = 0 \) for all \( i \), this contradicts the assumption that we started with a nontrivial linear dependence relation. Hence, \( S \) must be linearly independent. \( \square \)

The following theorem implies that in a finite dimensional inner product space (or Hermitian inner product space) with an orthogonal basis, it is easy to find the coordinates of a given vector in terms of this basis:

**Theorem 2.** Suppose that \( \mathcal{B} = \{v_1, v_2, \ldots, v_n\} \) is an orthogonal basis for \( V \), and \( v \in V \). Then,
\[
v = \frac{\langle v, v_1 \rangle}{\langle v_1, v_1 \rangle}v_1 + \frac{\langle v, v_2 \rangle}{\langle v_2, v_2 \rangle}v_2 + \ldots + \frac{\langle v, v_n \rangle}{\langle v_n, v_n \rangle}v_n.
\]

**Proof:** Since \( \mathcal{B} \) is a basis, there must exist (real or complex) constants \( c_1, c_2, \ldots, c_n \) such that
\[
v = c_1v_1 + c_2v_2 + \ldots + c_nv_n.
\]
Take the inner product of both sides by \( v_i \). This gives us
\[
\langle v, v_i \rangle = \langle c_1v_1 + \ldots + c_{i-1}v_{i-1} + c_iv_i + c_{i+1}v_{i+1} + \ldots + c_nv_n, v_i \rangle
\]
\[
\langle v, v_i \rangle = c_i\langle v_i, v_i \rangle
\]
\[
\frac{\langle v, v_i \rangle}{\langle v_i, v_i \rangle} = c_i
\]
which completes the proof. (Again, the linearity of inner product (or Hermitian inner product) in the first entry, and the orthogonality of \( \mathcal{B} \) is used in order to pass to line 2 from line 1.) \( \square \)

**Exercise:** Check that \( \mathcal{B} = \{(1, -2, 0), (2, 1, 0), (0, 0, 3)\} \) is an orthogonal basis for \( \mathbb{R}^3 \) with its standard inner product. Write the vector \((5, 7, 12)\) as a linear combination of elements of \( \mathcal{B} \) by using the formula in the preceding theorem.

## 2 Gram-Schmidt Orthogonalization Process

Let \( (V, \langle \cdot, \cdot \rangle) \) be an inner product space or a Hermitian inner product space and suppose that \( V \) is finite dimensional. A natural question is whether every such space admits an orthogonal basis. The following theorem guarantees that the answer is positive, furthermore it provides an efficient algorithm for finding such an orthogonal basis starting from any basis for \( V \).
Theorem 3. (Gram-Schmidt orthogonalization) Suppose that $(V, \langle \cdot, \cdot \rangle)$ is an $n$ dimensional inner product space or a Hermitian inner product space with a basis $B = \{v_1, v_2, \ldots, v_n\}$. Define $w_i$ for $i = 1, \ldots, n$ recursively by the formula:

\[
\begin{align*}
    w_1 &= v_1 \\
    w_2 &= v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 \\
    w_3 &= v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 \\
    &\vdots \\
    w_n &= v_n - \frac{\langle v_n, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_n, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_n, w_{n-1} \rangle}{\langle w_{n-1}, w_{n-1} \rangle} w_{n-1}
\end{align*}
\]

Then, $B' = \{w_1, w_2, \ldots, w_n\}$ is an orthogonal basis for $V$.

Proof: It will suffice to show that $\langle w_i, w_j \rangle = 0$ for any $i \neq j$ and that $w_i$ is not the zero vector for any $i$, since these would say that $B'$ is an orthogonal set and does not contain the zero vector. But then, by one of the theorems above, it is linearly independent. Since it contains exactly $n$ elements, it then must be a basis as well. To show these, first define

\[W_k = \text{Span}(w_1, w_2, \ldots, w_k)\]

for each $k \in \{1, \ldots, n\}$. We claim that $W_k = \text{Span}(v_1, v_2, \ldots, v_k)$. Let us prove this by induction. It is clear for $k = 1$. Suppose it holds for all $k = i - 1$, therefore $\text{Span}(v_1, v_2, \ldots, v_{i-1}) = \text{Span}(w_1, w_2, \ldots, w_{i-1})$. Now,

\[w_i = v_i - \frac{\langle v_i, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_i, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_i, w_{i-1} \rangle}{\langle w_{i-1}, w_{i-1} \rangle} w_{i-1} \in \text{Span}(v_1, v_2, \ldots, v_{i-1}) \subseteq \text{Span}(v_1, v_2, \ldots, v_i)\]

(the last inclusion is because of the inductive assumption). Conversely,

\[v_i = w_i + \frac{\langle v_i, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v_i, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \cdots + \frac{\langle v_i, w_{i-1} \rangle}{\langle w_{i-1}, w_{i-1} \rangle} w_{i-1} \subseteq \text{Span}(w_1, w_2, \ldots, w_i)\]

These two inclusions together imply that $\text{Span}(v_1, v_2, \ldots, v_i) = \text{Span}(w_1, w_2, \ldots, w_i)$, so the induction is complete.

The claim first of all implies that $w_k$ is not the zero vector for any $k$. Indeed, $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set, hence $W_k = \text{Span}(v_1, \ldots, v_k)$ must be $k$-dimensional. But if $w_k$ is the zero vector, $W_k = \text{Span}(w_1, \ldots, w_k)$ has dimension less than $k$, a contradiction.

Now, let us show that $\langle w_i, w_j \rangle = 0$ for any $i \neq j$. Without loss of generality, assume $i < j$. Let us do induction on $j$. It is clear for $j = 2$ (and $i = 1$) by direct computation. Now suppose the claim holds up to $j - 1$ for the larger index. Then

\[
\begin{align*}
    \langle w_j, w_i \rangle &= \langle v_j - \frac{\langle v_j, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_j, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \cdots - \frac{\langle v_j, w_{j-1} \rangle}{\langle w_{j-1}, w_{j-1} \rangle} w_{j-1}, w_i \rangle \\
    &= \langle v_j, w_i \rangle - \frac{\langle v_j, w_i \rangle}{\langle w_i, w_i \rangle} w_i \\
    &= 0
\end{align*}
\]
which finishes the proof (the transition from first line to second line is by linearity in the first entry and the inductive assumption). □

Exercise: Let $V = P_3(\mathbb{R})$ be the vector space of polynomials in one variable over $\mathbb{R}$ with degree $\leq 3$, equipped with the integral inner product

$$\langle p(x), q(x) \rangle = \int_0^1 p(x)q(x)dx$$

(a) Show that this is an inner product on $V$

(b) Starting from the basis $B = \{1, x, x^2, x^3\}$, apply the Gram-Schmidt orthogonalization process in order to find an orthogonal basis $B'$ for $V$. 
Orthogonal Projections

Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space, or a Hermitian inner product space. Suppose that \(v, w \in V\) and \(w\) is not the zero vector. We already defined the projection \(\text{proj}_w(v)\) of \(v\) along \(w\). We want to generalize this concept now, to the orthogonal projection of \(v\) to a subspace \(W\) of \(V\).

**Definition 1.** Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space, or a Hermitian inner product space. Let \(W\) be a subspace of \(V\) and \(v \in V\). An orthogonal projection of \(v\) along \(W\) is a vector \(u\) such that

- \(u \in W\)
- \(v - u \perp W\)

(Of course, \(v - u \perp W\) means \(v - u\) should be orthogonal to all vectors in \(W\).)

**Theorem 1.** Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space, or a Hermitian inner product space. Let \(W\) be a finite dimensional subspace of \(V\) and \(v \in V\). Then there exists a unique orthogonal projection of \(v\) along \(W\).

**Proof:** Existence: Since \(W\) is finite dimensional (say \(\dim(W) = m\)), it has an orthogonal basis \(w_1, w_2, \ldots, w_m\) by the Gram-Schmidt orthogonalization process. Let

\[
u = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \ldots + \frac{\langle v, w_m \rangle}{\langle w_m, w_m \rangle} w_m.
\]

We claim that \(u\) is an orthogonal projection of \(v\) along \(W\). It is clear that \(u \in W\). So it remains to check that \(v - u \perp W\). In order to show this, it is enough to check that \(v - u \perp w_i\) for every \(i \in \{1, 2, \ldots, m\}:

\[
\langle v - u, w_i \rangle = \left\langle v - \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 - \ldots - \frac{\langle v, w_m \rangle}{\langle w_m, w_m \rangle} w_m, w_i \right\rangle
= \langle v, w_i \rangle - \langle v, w_i \rangle \frac{\langle w_i, w_i \rangle}{\langle w_i, w_i \rangle} w_i
= 0.
\]

Hence the existence of an orthogonal projection is proven.

Uniqueness: Suppose that \(u_1, u_2\) are both orthogonal projections of \(v\) along \(W\). Then since \(u_1, u_2 \in W\), we have \(u_1 - u_2 \in W\). Since \(v - u_1 \perp W\) and \(v - u_2 \perp W\), by taking their difference, we get \(u_1 - u_2 \perp W\). But then \(u_1 - u_2 \perp u_1 - u_2\). This says

\[
\langle u_1 - u_2, u_1 - u_2 \rangle = ||u_1 - u_2||^2 = 0
\]

which happens iff \(u_1 = u_2\). Therefore, uniqueness is proven. \(\square\)
The theorem tells us that the orthogonal projection of \(v\) to a finite dimensional subspace \(W\) of \(V\) is a well defined notion. Let us denote it by \(\text{proj}_W(v)\). We obtained a formula during the proof: If \(w_1, w_2, \ldots, w_m\) is an orthogonal basis for \(W\), then

\[
\text{proj}_W(v) = \frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 + \ldots + \frac{\langle v, w_m \rangle}{\langle w_m, w_m \rangle} w_m.
\]

The next result tells us that orthogonal projections minimize the distance between the projected vector and vectors belonging to the subspace projected to. This is an extremely important fact for various applications.

**Theorem 2.** Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space, or a Hermitian inner product space. Let \(W\) be a finite dimensional subspace of \(V\) and \(v \in V\). Then for any \(w \in W\),

\[
||v - \text{proj}_W(v)|| \leq ||v - w||
\]

with equality if and only if \(w = \text{proj}_W(v)\).

**Proof:** Since the quantities in the inequality are nonnegative, it is enough to show that their squares satisfy the same inequality.

\[
||v - w||^2 = \langle v - w, v - w \rangle = \langle v - \text{proj}_W(v) + (\text{proj}_W(v) - w), v - \text{proj}_W(v) + (\text{proj}_W(v) - w) \rangle
\]

\[
= \langle v - \text{proj}_W(v), v - \text{proj}_W(v) \rangle + \langle v - \text{proj}_W(v), \text{proj}_W(v) - w \rangle + \langle \text{proj}_W(v) - w, v - \text{proj}_W(v) \rangle + \langle \text{proj}_W(v) - w, \text{proj}_W(v) - w \rangle
\]

\[
= \langle v - \text{proj}_W(v), v - \text{proj}_W(v) \rangle + \langle \text{proj}_W(v) - w, \text{proj}_W(v) - w \rangle
\]

\[
= ||v - \text{proj}_W(v)||^2 + ||\text{proj}_W(v) - w||^2
\]

\[
\geq ||v - \text{proj}_W(v)||^2.
\]

(In this derivation, the fact that \(\text{proj}_W(v) - w \in W\), therefore \(v - \text{proj}_W(v) \perp \text{proj}_W(v) - w\) was used.) This proves the inequality. Furthermore, equality holds if and only if \(||\text{proj}_W(v) - w|| = 0\), which happens if and only if \(w = \text{proj}_W(v)\). \(\Box\)

**Exercise:** Let \(\mathbb{R}^4\) be equipped with its standard inner product. Find the shortest distance between the vector \((1, 2, 3, 4)\) and the 2-dimensional subspace \(W\) spanned by \((1, -1, 1, -1)\) and \((1, 2, 0, 0)\).
1 Method of Least Squares

Suppose that we have an \( m \times n \) real or complex linear system, where typically \( m \) is much larger than \( n \):

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
  a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
  &\vdots \\
  a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m
\end{align*}
\]

If \( m \) is larger than \( n \) (the system is overdetermined), intuitively one does not expect to be able to find any solutions for \( x_1, \ldots, x_n \) unless something special happens. More precisely, by our discussions in MATH 261, the system has a solution if and only if the right hand side vector belongs to the span of the column vectors of the coefficient matrix \( A \), i.e.

\[
\text{System has a solution } \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \iff \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in W = \text{Span} \left( \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right).
\]

However, clearly \( \dim(W) \leq n \) since the coefficient matrix \( A \) has \( n \) columns. So if \( n < m \), the subspace \( W \) is strictly smaller than \( \mathbb{R}^m \) or \( \mathbb{C}^m \), so we shouldn’t expect to get a randomly chosen \( \mathbf{b} \in \mathbb{R}^m \) or \( \mathbb{C}^m \) to lie in \( W \).

Nevertheless, even if the system does not have a solution, in many applications it is of interest to find an approximate solution, namely a vector \( \mathbf{x} \) such that the quantity

\[
\| \mathbf{b} - A\mathbf{x} \|
\]

is minimized (an honest solution exists if and only if this quantity can be made 0). The norm is usually with respect to the standard inner product on \( \mathbb{R}^m \) or the standard Hermitian inner product on \( \mathbb{C}^m \), but other choices are equally valid and such a choice depends on the application in mind. But we know the answer to this question from our discussion of orthogonal projections:

\[
\| \mathbf{b} - A\mathbf{x} \| \text{ is minimized } \iff A\mathbf{x} = \text{proj}_W(\mathbf{b}).
\]

Hence we can outline now the method of least squares for finding an approximate solution to an overdetermined system \( A\mathbf{x} = \mathbf{b} \), given an inner product on \( \mathbb{R}^n \) or a Hermitian inner product on \( \mathbb{C}^n \):

\[
\text{Method of Least Squares}
\]

\[
\frac{1}{n} A^T A \mathbf{x} = A^T \mathbf{b}
\]

\[
\mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}
\]
1 METHOD OF LEAST SQUARES

• Compute an orthogonal basis for $W$,
• Find $\text{proj}_W(b)$,
• Solve $Ax = \text{proj}_W(b)$

Remark: One can carry this procedure without first finding an orthogonal basis for $W$, but then one has to invert a certain matrix instead.

Example: In a physics experiment, suppose it is theoretically known that a quantity $y$ should depend (affine) linearly on another quantity $x$, namely there exist real constants $a, b$ such that $y = ax + b$. Say the person doing the experiment is trying to find $a$ and $b$. Suppose that after running the experiment for a few times, the following data is collected:

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3.2</td>
</tr>
<tr>
<td>3</td>
<td>3.9</td>
</tr>
<tr>
<td>4</td>
<td>4.7</td>
</tr>
<tr>
<td>5</td>
<td>6.1</td>
</tr>
</tbody>
</table>

Of course, there may be some experimental errors, so one cannot find $a$ and $b$ that fits all lines of this table perfectly at once (try it! We see that it should be something around $y = x + 1$, but probably this is not the best that we can do). Instead, we think of this as an overdetermined system

\[
\begin{align*}
    a + b &= 2 \\
    2a + b &= 3.2 \\
    3a + b &= 3.9 \\
    4a + b &= 4.7 \\
    5a + b &= 6.1
\end{align*}
\]

This $5 \times 2$ linear system does not have any solutions. We will find an approximate solution by the method of least squares, taking the standard inner product on $\mathbb{R}^5$.

Step 1: According to our notation above,

\[
W = \text{Span} \begin{pmatrix} v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{pmatrix}.
\]

Let us find an orthogonal basis for $W$. By the Gram-Schmidt orthogonalization process, we can take

\[
w_1 = v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}.
\]
Next,
\[
 w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1
\]
\[
= \begin{bmatrix}
 1 \\
 1 \\
 1 \\
 1
\end{bmatrix}
- \frac{15}{55}
\begin{bmatrix}
 1 \\
 2 \\
 3 \\
 4 \\
 5
\end{bmatrix}
\]
\[
= \begin{bmatrix}
 8 \\
 5 \\
 2 \\
 -1 \\
 -4
\end{bmatrix}
= \frac{1}{11}
\begin{bmatrix}
 8 \\
 5 \\
 2 \\
 -1 \\
 -4
\end{bmatrix}
\]

Step 2: Compute \( \text{proj}_W(y) \) where
\[
y = \begin{bmatrix}
 2 \\
 3.2 \\
 3.9 \\
 4.7 \\
 6.1
\end{bmatrix}
\]

We will round the computations to two decimal places.
\[
\text{proj}_W(y) = \frac{\langle y, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 + \frac{\langle y, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2
\]
\[
= \frac{69.4}{55}
\begin{bmatrix}
 1 \\
 2 \\
 3 \\
 4 \\
 5
\end{bmatrix}
+ \frac{10.7}{11}
\begin{bmatrix}
 8 \\
 5 \\
 2 \\
 -1 \\
 -4
\end{bmatrix}
\]
\[
= \begin{bmatrix}
 2.04 \\
 3.01 \\
 3.98 \\
 4.95 \\
 5.92
\end{bmatrix}
\]

Step 3: Solve the new system \( Ax = \text{proj}_W(y) \). Explicitly, this system now reads:
\[
a + b = 2.04
\]
\[
2a + b = 3.01
\]
\[
3a + b = 3.98
\]
\[
4a + b = 4.95
\]
\[
5a + b = 5.92
\]

In comparison to the original system, the big difference is that the new system is consistent. One can easily solve for \( a, b \) and find
\[
a = 0.97 \quad b = 1.07
\]
Therefore, the “best line” that fits the experimental data is

\[ y = 0.97x + 1.07 \]

Of course, the notion of “best” here means distance minimizing with respect to the standard inner product on \( \mathbb{R}^5 \). If we chose another inner product, we would have another interpretation of “best”.
1 Fourier Series

Let $V = C[a,b]$ denote the set of continuous real valued functions on the interval $[a,b]$, viewed as a vector space over $\mathbb{R}$ with respect to the usual operations. Endow $V$ with the integral inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$ 

For many applications, it is important to find large subsets of $V$ which are orthogonal. There are two comments to make here: The largest subset that one can hope for is an orthogonal basis for $V$, since we know that orthogonal sets are linearly independent. However, $V$ is an infinite dimensional vector space so this is usually not practical; one would have to go into a set theoretical discussion which shifts the emphasis away from the application in mind; the functions constructed would be difficult (or impossible) to describe. Rather one uses linear algebra together with some analysis: It is of great interest to find an orthogonal set $S$, such that the closure of $\text{Span}(S)$ inside $V$ is $V$ itself (namely, $\text{Span}(S)$ is dense in $V$). Of course, one has to carefully define the metric on $V$ by using the inner product, and carefully define closure and related concepts, so this is not just linear algebra but we would be sailing towards metric spaces and functional analysis. We will not do this analysis part here, however, and just talk about the linear algebra part. Our goal will be to demonstrate a large orthogonal set. For the rest, please wait for a course in metric spaces, functional analysis or Fourier series.

Let us specifically take $a = -\pi$, $b = \pi$, therefore $V = C[-\pi, \pi]$. An extremely useful orthogonal set is the following:

$$S = \{ \sin(nx) | n = 1, 2, 3, \ldots \} \cup \{ \cos(nx) | n = 0, 1, 2, \ldots \}.$$ 

Proposition 1. The set $S$ above is an orthogonal set with respect to the integral inner product on $V$.

Proof: We just need to show that

- $\langle \sin(nx), \sin(mx) \rangle = 0$ if $n \neq m$,
- $\langle \cos(nx), \cos(mx) \rangle = 0$ if $n \neq m$,
- $\langle \sin(nx), \cos(mx) \rangle = 0$ for all $n, m$.

Let us start by showing the first identity. Suppose $n \neq m$. Then,

$$\langle \sin(nx), \sin(mx) \rangle = \int_{-\pi}^{\pi} \sin(nx) \sin(mx) dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (\cos((n-m)x) - \cos((n+m)x)) dx$$

$$= \frac{1}{2(n-m)} \sin((n-m)x) \bigg|_{-\pi}^{\pi} - \frac{1}{2(n+m)} \sin((n+m)x) \bigg|_{-\pi}^{\pi}$$

$$= 0.$$
Note that, \( n \neq m \) is used in a crucial way at the integration step since if \( n = m \), we get \( \cos((n - m)x) = \cos(0) = 1 \) and its integral would give a nonzero answer. Also, the fact that \( n, m \) are integers is important in the final evaluation in order to get 0.

The other two computations are quite similar. Say \( n \neq m \) again. Then,

\[
\langle \cos(nx), \cos(mx) \rangle = \frac{1}{2} \int_{-\pi}^{\pi} \cos((n - m)x) + \cos((n + m)x) dx
\]

\[
= \frac{1}{2} \left[ \frac{\sin((n - m)x)}{n - m} \bigg|_{-\pi}^{\pi} - \frac{\sin((n + m)x)}{2(n + m)} \bigg|_{-\pi}^{\pi} \right]
\]

\[= 0.\]

Again, it was important here that \( n \neq m \) and that they are both integers.

Finally, for any \( n \geq 1 \) and \( m \geq 0 \) integers (no condition about them being different this time),

\[
\langle \sin(nx), \cos(mx) \rangle = \int_{-\pi}^{\pi} \sin(nx) \cos(mx) dx
\]

\[= 0.\]

It is a little bit easier to see the equality this time: Notice that \( \sin(nx) \cos(mx) \) is an odd function, therefore its integral over the symmetric interval \([-\pi, \pi]\) is 0. We did not need anything about \( n, m \) not being equal (and in fact we don’t need them to be integers either). This finishes the proof of the proposition. \( \square \)

Now, let \( W = \text{Span}(S) \). We would like to calculate the orthogonal projection of an element \( f \in V \) along \( W \). The technical problem is that \( W \) is infinite dimensional, whereas we only discussed orthogonal projections along finite dimensional subspaces. Instead, let us define finite subsets \( S_k \) of \( S \):

\[ S = \{ \sin(nx) | n = 1, 2, 3, \ldots, k \} \cup \{ \cos(nx) | n = 0, 1, 2, \ldots, k \}, \]

and let \( W_k = \text{Span}(S_k) \). Now, clearly \( W_k \) is finite dimensional and \( S_k \) is an orthogonal basis for it. For any \( f \in V \), we can consider its projection \( \text{proj}_{W_k}(f) \). By the projection formula that we obtained before,

\[
\text{proj}_{W_k}(f) = \sum_{n=1}^{k} \frac{\langle f, \sin(nx) \rangle}{\sin(nx), \sin(nx)} \sin(nx) + \sum_{n=0}^{k} \frac{\langle f, \cos(nx) \rangle}{\cos(nx), \cos(nx)} \cos(nx).
\]

We can compute the coefficients in the formula as follows: It is clear by definition that

\[
\langle f, \sin(nx) \rangle = \int_{-\pi}^{\pi} f(x) \sin(nx) dx, \quad \langle f, \cos(nx) \rangle = \int_{-\pi}^{\pi} f(x) \cos(nx) dx.
\]

On the other hand,

**Lemma 1.** We have the identities:
1 FOURIER SERIES

• \((\sin(nx), \sin(nx)) = \pi\) for all integers \(n \geq 1\),
• \((\cos(nx), \cos(nx)) = \pi\) for all integers \(n \geq 1\),
• \((\cos(nx), \cos(nx)) = 2\pi\) for \(n = 0\).

**Proof:** Exercise. □

Altogether, this gives us the proof of the following proposition:

**Proposition 2.** The projection of any \(f \in V\) along \(W_k = \text{Span}(S_k)\) can be computed to be

\[
\text{proj}_{W_k}(f) = \sum_{n=1}^{k} a_n \sin(nx) + \sum_{n=0}^{k} b_n \cos(nx)
\]

where the coefficients \(a_n\) and \(b_n\) can be obtained by using the formulas

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx, \quad n \geq 1
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \quad n \geq 1
\]

\[
b_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx
\]

It is natural to ask what happens when we take \(k \to +\infty\). Instead of trying to define a projection along \(W\) (which is infinite dimensional), we can simply take \(k\) to infinity in the formula above. This brings us to the following definition

**Definition 1.** Let \(f \in V\). Its **Fourier series** is defined to be the following infinite series:

\[
\sum_{n=1}^{\infty} a_n \sin(nx) + \sum_{n=0}^{\infty} b_n \cos(nx)
\]

where the **Fourier coefficients** \(a_n\) and \(b_n\) are given by the formulas in the previous proposition.

A priori, it is unclear whether this infinite series will converge or not for any given value of \(x\), and if so, to what value it will converge. But the wonderful observation of Fourier below gives a complete answer to this question. We will not prove the following theorem in this course, since it requires some analysis (although it is not very difficult):

**Theorem 1.** (Fourier convergence theorem) Let \(f \in V = C[-\pi, \pi]\). Then the Fourier series of \(f\) converges for all \(x\) to the function \(f\).

**Exercise:** Try to formulate analogous results to those in this lecture up to the last theorem with \(V\) replaced by the set of complex valued continuous functions on \([-\pi, \pi]\), the integral inner product replaced by its Hermitian counterpart, and the set \(S\) replaced by

\[
S = \{e^{inx} | n \in \mathbb{Z}\}.
\]

Then, look up the statement of Fourier convergence theorem in this setup from the literature.
1 Self Adjoint Operators

Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space, or a Hermitian inner product space. Let \(T : V \to V\) be a linear operator. A linear operator \(T^* : V \to V\) such that
\[
\langle Tv, w \rangle = \langle v, T^*w \rangle
\]
for all \(v, w \in V\) is called an \textit{adjoint} of \(T\). We will show that in the case \(V\) is finite dimensional, the adjoint of \(T\) always exists and is unique. Furthermore, operators satisfying \(T = T^*\) are called \textit{self-adjoint}. Such operators are extremely important and enjoy very nice properties which will be studied in the forthcoming lectures.

1.1 Riesz Representation Theorem

\textbf{Theorem 1.} Let \((V, \langle \cdot, \cdot \rangle)\) be an inner product space, or a Hermitian inner product space, where \(V\) is finite dimensional. Let \(\varphi\) be a linear functional on \(V\). Then there exists a unique \(u \in V\) such that
\[
\varphi(v) = \langle v, u \rangle
\]
for all \(v \in V\).

\textit{Proof:} Existence: By the Gram-Schmidt orthogonalization process, we know that \(V\) has an orthonormal basis \(\{u_1, u_2, \ldots, u_n\}\). We can use the simple fact that \(v = proj_V(v)\) and the orthogonal projection formula that we obtained before: Together with the fact that \(\langle u_i, u_i \rangle = 1\) for an orthonormal set of vectors, this gives us
\[
v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \ldots + \langle v, u_n \rangle u_n.
\]
Now, since \(\varphi\) is linear, it is clear that
\[
\varphi(v) = \langle v, u_1 \rangle \varphi(u_1) + \langle v, u_2 \rangle \varphi(u_2) + \ldots + \langle v, u_n \rangle \varphi(u_n).
\]
By looking at this expression carefully, we can construct the vector \(u\) manually: Let us do the Hermitian case; in the real case the formula that we will write will also hold in particular. Let
\[
u = \overline{\varphi(u_1)} u_1 + \overline{\varphi(u_2)} u_2 + \ldots + \overline{\varphi(u_n)} u_n.
\]
Let us check that \(u\) really satisfies the equality \(\varphi(v) = \langle v, u \rangle\) for all \(v\), which will finish the existence part of the proof:
\[
\langle v, u \rangle = \langle v, \overline{\varphi(u_1)} u_1 + \overline{\varphi(u_2)} u_2 + \ldots + \overline{\varphi(u_n)} u_n \rangle
\]
\[
= \langle v, \overline{\varphi(u_1)} u_1 \rangle + \langle v, \overline{\varphi(u_2)} u_2 \rangle + \ldots + \langle v, \overline{\varphi(u_n)} u_n \rangle
\]
\[
= \overline{\varphi(u_1)} \langle v, u_1 \rangle + \overline{\varphi(u_2)} \langle v, u_2 \rangle + \ldots + \overline{\varphi(u_n)} \langle v, u_n \rangle
\]
\[
= \varphi(u_1) \langle v, u_1 \rangle + \varphi(u_2) \langle v, u_2 \rangle + \ldots + \varphi(u_n) \langle v, u_n \rangle
\]
\[
= \varphi(v)
\]
Uniqueness: Suppose that $\varphi(v) = \langle v, u \rangle = \langle v, w \rangle$ for all $v \in V$. We want to show that $u = w$. Subtracting the two equations, we get

$$\langle v, u - w \rangle = 0$$

for all $v \in V$. Now, we can take $v = u - w$ in particular, and get

$$\langle u - w, u - w \rangle = ||u - w||^2 = 0.$$ 

But this immediately implies that $u - w = 0$, namely $u = w$. This finishes the proof. \hfill $\square$

1.2 Adjoint of an Operator

**Definition 1.** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, or a Hermitian inner product space. Suppose that $T : V \to V$ is a linear operator. A linear operator $T^* : V \to V$ is called an **adjoint** of $T$ if

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for all $v, w \in V$.

**Theorem 2.** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, or a Hermitian inner product space, and assume that $V$ is finite dimensional. Then every linear operator $T : V \to V$ has a unique adjoint $T^*$.

**Proof:** Existence: Let us first define the map $T^*$ for a given operator $T$. Given any $w \in V$, we claim that the map $\varphi$ given by the formula

$$\varphi(v) = \langle Tv, w \rangle$$

is a linear functional on $V$. Indeed,

$$\varphi(c_1v_1 + c_2v_2) = \langle T(c_1v_1 + c_2v_2), w \rangle$$

$$= \langle c_1Tv_1 + c_2Tv_2, w \rangle$$

$$= c_1\langle Tv_1, w \rangle + c_2\langle Tv_2, w \rangle$$

$$= c_1\varphi(v_1) + c_2\varphi(v_2)$$

so $\varphi$ is a linear functional. Next, we can apply the Riesz Representation Theorem to the functional $\varphi$, which tells us that there exists a unique vector $u \in V$ such that

$$\varphi(v) = \langle Tv, w \rangle = \langle v, u \rangle.$$ 

What we must be careful about here is that $u$ depends on $w$, which we fixed in the beginning. So we will set $u = T^*w$, which defines the map $T^* : V \to V$. Now, for all $v, w \in V$ we have

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

Next, we need to prove that $T^*$ is a linear operator:

$$\langle v, T^*(c_1w_1 + c_2w_2) \rangle = \langle Tv, c_1w_1 + c_2w_2 \rangle$$

$$= \overline{c_1}\langle Tv, w_1 \rangle + \overline{c_2}\langle Tv, w_2 \rangle$$

$$= \overline{c_1}\langle v, T^*w_1 \rangle + \overline{c_2}\langle v, T^*w_2 \rangle$$

$$= \langle v, c_1T^*w_1 + c_2T^*w_2 \rangle.$$
But again, since this equality holds for all \( v \), we have
\[
T^*(c_1w_1 + c_2w_2) = c_1T^*w_1 + c_2T^*w_2
\]
which implies that \( T^* \) is a linear operator.

Uniqueness: The uniqueness of \( T^* \) easily follows from the uniqueness statement in the Riesz Representation Theorem. With the notation above, for every \( w \in V \), there exists a unique vector \( T^*w \in V \) such that
\[
\varphi(v) = \langle Tv, w \rangle = \langle v, T^*w \rangle
\]
for all \( v \in V \). This shows that \( T^* \) is uniquely defined and finishes the proof. \( \square \)

Example: Suppose that \( V = \mathbb{R}^n \) with its standard inner product. If we identify each \( v \in V \) by an \( n \times 1 \) column vector, which is again denoted by \( v \), it is clear that
\[
\langle v, w \rangle = v^T w.
\]
Let us give a quick proof: Suppose that
\[
v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad w = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.
\]
Then, \( v^T w \) will be a \( 1 \times 1 \) matrix, hence it can be identified by the value of its entry in \( \mathbb{R} \), and we have
\[
v^T w = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \ldots + a_nb_n = \langle v, w \rangle.
\]
Now, let \( T : V \to V \) be a linear operator. Then \( T \) can be represented with respect to the standard basis by an \( n \times n \) matrix \( A \). Also denoting the matrix corresponding to the adjoint operator \( T^* \) by \( A^* \), the equality \( \langle Tv, w \rangle = \langle v, T^*w \rangle \) implies
\[
(Av)^T w = v^T (A^*w), \quad \forall v, w \in V
\]
\[
v^T A^T w = v^T A^*w, \quad \forall v, w \in V
\]
\[
\Rightarrow A^* = A^T
\]
Therefore, in this setup, the matrix representing \( T^* \) is simply the transpose of the matrix representing \( T \).

Example: This time, suppose that \( V = \mathbb{C}^n \) with its standard Hermitian product. Let us repeat what we did in the previous example in this case. Again identify each \( v \in V \) with an \( n \times 1 \)
column vector, this time with complex entries. This time, if $v$ and $w$ are as above, then we have

$$\langle v, w \rangle = a_1 \bar{b}_1 + a_2 \bar{b}_2 + \ldots + a_n \bar{b}_n$$

$$= \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_n \end{bmatrix}$$

$$= v^T w$$

Now, if $A$ represents $T$ and $A^*$ represents $T^*$ with respect to the standard basis, then we get

$$(Av)^T w = v^T (A^T w), \ \forall v, w \in V$$

$$v^T A^T w = v^T \bar{A}^T w, \ \forall v, w \in V$$

$$\Rightarrow A^* = \bar{A}^T$$

Therefore in this setup, the matrix representing $T^*$ is the conjugate transpose of $A$. Notice that this agrees with the previous example if the matrix $A$ is real. For this reason, sometimes in the context of matrices, $A^*$ is directly used to denote the conjugate transpose of a complex matrix $A$.

1.3 Self-Adjoint Operators

**Definition 2.** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, or a Hermitian inner product space. Suppose that $T : V \rightarrow V$ is a linear operator and $T^*$ is adjoint to $T$. If $T = T^*$, then $T$ is called a **self-adjoint operator**.

Notice that $T$ is a self-adjoint operator if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle, \ \forall v, w \in V.$$  

**Example:** Let $V = \mathbb{R}^n$ with its standard inner product. Let $T : V \rightarrow V$ be a linear operator and $A$ the $n \times n$ matrix representing $T$ with respect to the standard basis. Then, by our observations above, $T$ is self-adjoint if and only if $A = A^T$, namely if and only if $A$ is a **symmetric matrix**. This gives some intrinsic meaning for a matrix to be symmetric (apart from an aesthetic one).

If, instead we took $V = \mathbb{C}^n$ with its standard Hermitian inner product, and also $T$ and $A$ as above, this time $T$ is self-adjoint if and only if $A = A^* = A^T$. This coincides with the symmetric matrices in the real case, but has a name of its own:

**Definition 3.** An $n \times n$ complex matrix is called **Hermitian** if

$$A = \bar{A}^T.$$  

Then, a Hermitian matrix which has real entries is necessarily symmetric. But a symmetric complex matrix need not be Hermitian, these are two different notions.

**Exercises:**

1. Show that if $T_1, T_2$ are linear operators on an inner product space or a Hermitian inner product space, then $(T_1 T_2)^* = T_2^* T_1^*$.  

2. Show that the product of two symmetric matrices need not be symmetric, and the product of two Hermitian matrices need not be Hermitian. What if the matrices in question commute?

3. Show that linear combinations of self-adjoint operators on a real inner product space are self adjoint. Deduce that symmetric matrices form a subspace of $\text{Mat}_{n \times n}(\mathbb{R})$ (which is of course easy to see directly). What happens if you try to imitate these results for a Hermitian inner product space and for Hermitian matrices?
ORTHOGONAL AND UNITARY OPERATORS

1 ORTHOGONAL AND UNITARY OPERATORS

1 Orthogonal and Unitary Operators

Once we have the notion of the lengths of vectors defined by using an inner product on a real
vector space $V$, a natural question is “What are all linear operators on $V$ that preserve the
lengths of all vectors?” Before trying to answer this question, let us look at a motivational
element: Consider $V = \mathbb{R}^2$ with its standard inner product. We already know some linear
operators on $\mathbb{R}^2$ which preserve lengths of all vectors:

- Rotations about the origin, given by matrices of the form
  \[
  \begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
  \end{bmatrix}
  \]
  with respect to the standard basis,

- Reflections with respect to a line through the origin (making an angle of $\theta/2$ with the
  $x$-axis), given by matrices of the form
  \[
  \begin{bmatrix}
  \cos \theta & \sin \theta \\
  \sin \theta & -\cos \theta
  \end{bmatrix}.
  \]

It is a nice fact that these are actually all the linear operators on $\mathbb{R}^2$ which preserve the lengths
of all vectors, with respect to the standard inner product. You can try to give a direct geometric
proof. We will give a proof after establishing some results in much greater generality. Let us
now turn to the general setup.

1.1 Orthogonal Operators

Definition 1. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional, real inner product space. A linear operator
$T : V \to V$ is said to be orthogonal if

\[
\langle T v, T w \rangle = \langle v, w \rangle
\]

for all $v, w \in V$.

Proposition 1. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional, real inner product space and $T : V \to V$ is a linear operator. The following are equivalent:

1. $T$ is orthogonal.
2. $T$ preserves the lengths of all vectors
3. $TT^* = T^*T = I$ where $T^*$ denotes the adjoint of $T$.
4. $T$ takes an orthonormal basis for $V$ to an orthonormal basis.

Proof: (1 $\iff$ 2:) It is clear that if $T$ is orthogonal, then $\|Tv\|^2 = \langle T v, T v \rangle = \langle v, v \rangle = \|v\|^2$ for all
$v$, hence $T$ preserves the lengths of all vectors. Conversely, suppose that $T$ preserves the lengths
of all vectors. Observe that

\[
\|v + w\|^2 = \langle v + w, v + w \rangle = \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle = \|v\|^2 + 2\langle v, w \rangle + \|w\|^2.
\]

By using this relation, we can express the inner product purely in terms of lengths:

\[
\langle v, w \rangle = \frac{1}{2}(\|v + w\|^2 - \|v\|^2 - \|w\|^2).
\]
Then it becomes clear that, if a linear operator preserves the lengths of all vectors, then it must preserve the inner product, hence it must be orthogonal.

(1 $\iff$ 3:) Notice that, by definition of an adjoint operator $\langle Tv, Tw \rangle = \langle v, T^*Tw \rangle$. Now if $TT^* = T^*T = I$, then it is clear that

$$\langle Tv, Tw \rangle = \langle v, T^*Tw \rangle = \langle v, w \rangle$$

for all $v, w \in V$, hence $T$ is orthogonal. Conversely, if $T$ is orthogonal, then $\langle v, T^*Tw \rangle = \langle v, w \rangle$ for all $v, w \in V$. Subtracting the two sides we get

$$\langle v, (T^*T - I)w \rangle = 0.$$ 

Now, setting $v = (T^*T - I)w$ gives $||(T^*T - I)w||^2 = \langle (T^*T - I)w, (T^*T - I)w \rangle = 0$. This implies $(T^*T - I)w = 0$. But since this holds for all $w \in V$, we deduce that $T^*T = I$. Now, a left inverse to an operator on a finite dimensional vector space is also a right inverse. Hence $TT^* = I$ too.

(1 $\iff$ 4:) Suppose that $T$ is orthogonal and $B = \{v_1, \ldots, v_n\}$ is an orthonormal basis for $V$. This means, $\langle v_i, v_j \rangle = \delta_{ij}$ where $\delta_{ij}$ is equal to 1 if $i = j$ and 0 otherwise. Now,

$$\langle Tv_i, Tv_j \rangle = \langle v_i, v_j \rangle = \delta_{ij}.$$ 

This implies that $B' = \{Tv_1, \ldots, Tv_n\}$ is an orthonormal set. But we know that any orthonormal set is linearly independent, hence $B'$ is linearly independent. By (3), we know that $T^*T = I$, so $T$ is invertible, in particular it is injective. This tells us that we also have $|B'| = n = \dim(V)$, hence it must be a basis for $V$.

Conversely, suppose that $T$ takes an orthonormal basis $B = \{v_1, \ldots, v_n\}$ to an orthonormal basis $B' = \{Tv_1, \ldots, Tv_n\}$. Take any $v, w \in V$. Suppose $v = \sum a_i v_i$ and $w = \sum b_i v_i$. Then,

$$\langle Tv, Tw \rangle = \langle T(\sum a_i v_i), T(\sum b_i v_i) \rangle$$
$$= \langle \sum a_i Tv_i, \sum b_i Tv_i \rangle$$
$$= \sum_{i,j} a_i b_j \langle Tv_i, Tv_j \rangle$$
$$= \sum_{i,j} a_i b_j \delta_{ij}$$
$$= \sum_i a_i b_i$$
$$= \langle v, w \rangle$$

Hence the operator $T$ is orthogonal. This finishes the proof. □

**Example:** Let $V = \mathbb{R}^n$ and $\langle , \rangle$ is the standard inner product on $V$. Represent linear operators with respect to the standard basis by $n \times n$ matrices. In this case, we know from the previous lecture that, if the matrix corresponding to $T$ is $A$, then the matrix corresponding to $T^*$ is $A^T$, the transpose of $A$. In view of this, condition (3) in the above proposition becomes

$$AA^T = A^TA = I.$$ 

This leads us to make the following definition:
Definition 2. An $n \times n$ real matrix is called an **orthogonal matrix** if $AA^T = A^TA = I$.

**Exercise:** Let $A$ be an $n \times n$ real matrix.

1. Show that $A$ is an orthogonal matrix if and only if its $n$ columns form an orthonormal set with respect to the standard inner product.

2. Show that $A$ is an orthogonal matrix if and only if its $n$ rows form an orthonormal set with respect to the standard inner product.

3. Show that if $A$ is an orthogonal matrix then $\det(A) = \pm 1$.

4. Show that the set of all orthogonal matrices in $M_{n \times n}(\mathbb{R})$ form a group under matrix multiplication.

5. What are all $1 \times 1$ orthogonal matrices? What about $2 \times 2$?

1.2 **Unitary Operators**

Let us now define the analogous notions in the Hermitian setting.

**Definition 3.** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional, complex, Hermitian inner product space. A linear operator $T : V \to V$ is said to be **unitary** if

$$\langle Tv, Tw \rangle = \langle v, w \rangle$$

for all $v, w \in V$.

**Proposition 2.** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional, complex, Hermitian inner product space and $T : V \to V$ is a linear operator. The following are equivalent:

1. $T$ is unitary.

2. $T$ preserves the lengths of all vectors

3. $TT^* = T^*T = I$ where $T^*$ denotes the adjoint of $T$.

4. $T$ takes an orthonormal basis for $V$ to an orthonormal basis.

**Proof:** Exercise. Try to imitate the proof for the real case. There will be one or two tricky points: For instance $\langle w, v \rangle = \overline{\langle v, w \rangle}$ so one needs to upgrade the argument a little bit at that step somehow. Also, $\sum a_i b_j$ will be replaced by $\sum a_i \overline{b_j}$ etc. $\square$

**Example:** Let $V = \mathbb{C}^n$ and $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product on $V$. Represent linear operators with respect to the standard basis by $n \times n$ matrices. This time we know that, if the matrix corresponding to $T$ is $A$, then the matrix corresponding to $T^*$ is $A^* = \overline{A^T}$, the conjugate transpose of $A$. Therefore, condition (3) in the above proposition becomes

$$AA^* = A^*A = I.$$ 

Again, such matrices have a special name:

**Definition 4.** An $n \times n$ complex matrix is called a **unitary matrix** if $AA^* = A^*A = I$. 

Exercise: Let $A$ be an $n \times n$ complex matrix.

1. Show that $A$ is a unitary matrix if and only if its $n$ columns form an orthonormal set with respect to the standard Hermitian inner product.

2. Show that $A$ is a unitary matrix if and only if its $n$ rows from an orthonormal set with respect to the standard Hermitian inner product.

3. Show that if $A$ is a unitary matrix then $|\det(A)| = 1$ (notice that $\det(A)$ is a complex number now).

4. Show that the set of all unitary matrices in $M_{n\times n}(\mathbb{C})$ form a group under matrix multiplication.

5. What are all $1 \times 1$ unitary matrices? What about $2 \times 2$?
1 Normal Operators

So far, we have seen three special classes of operators (self-adjoint, orthogonal, unitary) whose definitions make use of the adjoint operator. All of these operators fall into a broader class, which has nice properties.

**Definition 1.** Let \((V, \langle \cdot, \cdot \rangle)\) be a finite dimensional inner product space, or a Hermitian inner product space. A linear operator \(T : V \rightarrow V\) is called a **normal operator** if

\[
TT^* = T^*T
\]

(in other words, if \(T\) commutes with its adjoint).

**Lemma 1.** Let \((V, \langle \cdot, \cdot \rangle)\) be a finite dimensional inner product space, or a Hermitian inner product space. Let \(T, S : V \rightarrow V\) be any linear operators. Then

1. \((T^*)^* = T\).
2. \((TS)^* = S^*T^*\).

**Proof:** (1) Let us do the Hermitian case. The real case follows with the same proof. Let \(v, w \in V\). Then,

\[
\langle Tv, w \rangle = \langle v, T^*w \rangle = \overline{\langle T^*w, v \rangle} = \overline{\langle w, (T^*)^*v \rangle} = \langle (T^*)^*v, w \rangle
\]

Now, subtracting the first and last expressions, we get \(\langle (T - (T^*)^*)v, w \rangle = 0\). By taking \(w = (T - (T^*)^*)v\) in particular, we see that \(||(T - (T^*)^*)v||^2 = 0\), so we must have \((T - (T^*)^*)v = 0\). But this holds for all \(v \in V\), therefore \(T - (T^*)^* = 0\), in other words \(T = (T^*)^*\).

(2) Again, take \(v, w \in V\). Then, \(\langle (TS)(v), w \rangle = \langle v, (TS)^*(w) \rangle\) by definition. On the other hand,

\[
\langle (TS)(v), w \rangle = \langle T(S(v)), w \rangle = \langle S(v), T^*(w) \rangle = \langle v, S^*(T^*(w)) \rangle
\]

Now, subtracting the two different expressions for \(\langle (TS)(v), w \rangle\) from each other, we get \(\langle v, ((TS)^* - S^*T^*)w \rangle = 0\) for all \(v, w \in V\). In particular, taking \(v = ((TS)^* - S^*T^*)w\) we get that \(((TS)^* - S^*T^*)w = 0\). Since this is true for all \(w\), we get \((TS)^* - S^*T^* = 0\), namely \((TS)^* = S^*T^*\). \(\square\)

**Proposition 1.** Self-adjoint operators, orthogonal operators and unitary operators are normal.

**Proof:** If \(T\) is self-adjoint, then \(T = T^*\). But then it is clear that \(TT^* = T^2 = T^*T\), therefore self-adjoint operators are normal. If \(T\) is orthogonal or unitary, then \(T^*T = TT^* = I\), in particular \(T^*T = TT^*\) so again \(T\) is normal.
2 Positive Operators

The following definition is somehow the correct analogue of a “positive real number” in the setup of linear operators:

**Definition 2.** Let \((V, \langle \cdot, \cdot \rangle)\) be a finite dimensional inner product space, or a Hermitian inner product space. A linear operator \(T : V \to V\) is called a **positive operator** if

1. \(T\) is self-adjoint,
2. \(\langle Tv, v \rangle\) is a positive real number for every nonzero \(v \in V\).

As a simple example, if \(T = cI\) where \(c\) is a positive real number, then one can immediately check that \(T\) is a positive operator.

**Proposition 2.** Let \((V, \langle \cdot, \cdot \rangle)\) be a finite dimensional inner product space, or a Hermitian inner product space. Let \(S, T\) be positive operators on \(V\). Then

1. \(S + T\) is a positive operator.
2. \(cT\) is a positive operator for any positive real number \(c\).

**Proof:**

(1) It is easy to check that \((S + T)^* = S^* + T^*\). Since \(S, T\) are self-adjoint, this implies that \(S + T\) is self-adjoint. Now, for any nonzero \(v \in V\),

\[
\langle (S + T)v, v \rangle = \langle Sv, v \rangle + \langle Tv, v \rangle > 0,
\]

therefore \(S + T\) is a positive operator.

(2) Again, it is easy to check that \((cT)^* = cT^*\) where \(c\) is a constant. In particular if \(c\) is real, then \((cT)^* = cT^*\). So if \(T\) is self-adjoint, then \(cT\) is also self adjoint for \(c\) real. Now if \(c > 0\), then for any nonzero \(v \in V\),

\[
\langle (cT)v, v \rangle = c\langle Tv, v \rangle > 0
\]

which shows that \(cT\) is positive. \(\square\)

**Caution:** An arbitrary linear combination of two positive operators is not necessarily positive. For example, if we multiply a positive operator with a negative real number, it is never positive. Therefore positive linear operators do not form a linear subspace of \(\text{Hom}(V, V)\). Rather, the set of positive linear operators is closed under addition, and under scalar multiplication with positive real numbers. It forms a so called **open cone** in \(\text{Hom}(V, V)\). \(\square\)

**Proposition 3.** Let \((V, \langle \cdot, \cdot \rangle)\) be a finite dimensional inner product space, or a Hermitian inner product space. Suppose that \(T\) is any **invertible** linear operator on \(V\). Then the operator \(T^*T\) and \(TT^*\) are positive operators.

**Proof:** First of all, let us show that \(TT^*\) is self-adjoint. Indeed,

\[
(TT^*)^* = (T^*)^*T^* = TT^*
\]
hence $TT^*$ is self-adjoint. A similar argument shows that $TT^*$ is also self-adjoint operator. To show positivity, suppose that $v \in V$ is a non-zero vector. Then,

$$\langle T^*Tv, v \rangle = \langle Tv, (T^*)^*v \rangle$$

$$= \langle Tv, Tv \rangle$$

$$\geq 0$$

Furthermore, inequality can happen iff $Tv = 0$. But since $T$ is invertible and $v$ is non-zero, this doesn’t happen. Hence $T^*T$ is a positive operator. To apply a similar argument for $TT^*$, it suffices to show that $T^*$ is invertible when $T$ is invertible. Suppose not. Then there exists non-zero $w \in V$ such that $T^*w = 0$. We can write $w = Tv$ for some $v \in V$ since $T$ is invertible. Then,

$$\langle w, w \rangle = \langle Tv, w \rangle$$

$$= \langle v, T^*w \rangle$$

$$= \langle v, 0 \rangle$$

$$= 0$$

But $w \neq 0$, so this is a contradiction. This finishes the proof. $\square$

**Remark:** This proposition should be seen as an analogue for linear operators of the fact that “the square of any nonzero real number is a positive number”.

2 \hspace{1cm} \textit{POSITIVE OPERATORS} \hspace{1cm} 3
MATHEMATICAL PAYMENTS

1 Spectral Theorem

The goal of this lecture is to prove important structural theorems about normal operators on a finite dimensional inner product space or a Hermitian inner product space. As opposed to arbitrary linear operators, normal linear operators are always diagonalizable.

Lemma 1. Let \((V, \langle \cdot, \cdot \rangle)\) be a finite dimensional inner product space (resp. Hermitian inner product space). Suppose that \(T\) is a normal operator on \(V\). Then

(a) for any polynomial \(f \in \mathbb{R}[x]\) (resp. \(f \in \mathbb{C}[x]\)), the operator \(f(T)\) is normal,
(b) if \(T\) is nilpotent, then \(T = 0\).

Proof: (a) Since \(T\) is normal, \(TT^* = T^*T\). To prove the claim, we need to show that \(f(T)f(T)^* = f(T)^*f(T)\). But this is straightforward, recalling the identities \((ST)^* = T^*S^*\) and \((c_1S + c_2T)^* = c_1S^* + c_2T^*\).

(b) As a preliminary step, let us show that \(T^2 = 0\) implies that \(T = 0\). Indeed, suppose \(T^2 = 0\). Then \((TT^*)^2 = T^2(T^*)^2 = 0\) (the first equality holds since \(T\) is normal). But now, for any \(v \in V\),

\[
0 = \langle 0, v \rangle = \langle (TT^*)^2v, v \rangle = \langle TT^*v, (TT^*)^*v \rangle = \langle TT^*v, TT^*v \rangle = ||TT^*v||^2
\]

But this immediately implies that \(TT^*v = T^*Tv = 0\) for all \(v \in V\). This in turn gives us \(0 = \langle T^*Tv, v \rangle = \langle Tv, Tv \rangle = ||Tv||^2\), so \(Tv = 0\) for all \(v\) and finally \(T = 0\), settling the claim.

Now, suppose that \(T^k = 0\) for some \(k \geq 2\) but \(T \neq 0\). Then, clearly \(T^{2^m} = 0\) whenever \(2^m \geq k\). Let \(s\) be the smallest positive integer such that \(T^{2^s} = 0\). By part (a), \(T^{2^{s-1}}\) is a normal operator. Since its square is 0, by the preliminary step in the previous paragraph, \(T^{2^{s-1}} = 0\). This contradicts the minimality of \(s\). Hence \(T = 0\) \(\square\)

Theorem 1. Every normal operator \(T\) on a finite dimensional inner product space or Hermitian inner product space \((V, \langle \cdot, \cdot \rangle)\) is diagonalizable over \(\mathbb{C}\).

Proof: Suppose that \(T\) is a normal operator on \(V\). By the fundamental theorem of algebra, its characteristic polynomial can be written as a product of linear factors

\[
\Delta_T(x) = (x - \lambda_1)^{n_1}(x - \lambda_2)^{n_2} \ldots (x - \lambda_k)^{n_k}
\]

where \(\lambda_1, \ldots, \lambda_k \in \mathbb{C}\) are the distinct eigenvalues of \(T\) and \(n_1 + \ldots + n_k = \dim(V)\). Let

\[
f(x) = (x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_k).
\]

We claim that the minimal polynomial \(\delta_T(x)\) of \(T\) is equal to \(f(x)\). Since \(\Delta_T(x)|\delta_T(x)|^n\), this implies that \(f(x)|\delta_T(x)\). Conversely, setting \(m = \max(n_1, \ldots, n_k)\) we see that

\[
f(T)^m = 0
\]
since $\Delta_T(x) f(x)^m$ and $\Delta_T(T) = 0$ by Cayley-Hamilton theorem. But then, $f(T)$ is normal by part (a) of the previous lemma and it is nilpotent. So by part (b) of the previous lemma, $f(T) = 0$. This implies that $\delta_T(x) f(x)$ and consequently $\delta_T(x) = f(x)$. Finally, $T$ is diagonalizable since one of the characterizations of diagonalizability of an operator is that its minimal polynomial is a product of distinct linear factors. □

Remark: The careful reader is probably already disturbed at this point by the term “$T$ is diagonalizable over $\mathbb{C}$” for the case when $V$ is a real vector space. After all, $V$ is a real vector space. What does it mean for $T$ to have a basis of complex eigenvectors (which cannot belong to $V$)? This is a valid technical objection which can be quickly resolved as follows: If $V$ is a real vector space, the theorem is valid after replacing $V$ by its complexification: $V_\mathbb{C} = V \otimes \mathbb{C}$. This is a tensor product of two real vector spaces. Its real dimension is $2 \dim(V)$. But it also admits a $\mathbb{C}$-vector space structure in the obvious way, and its complex dimension is equal to $\dim(V)$. The precise statement of the above theorem is that $V_\mathbb{C}$, viewed as a $\mathbb{C}$-vector space, has a basis of eigenvectors of $T$. This technical correction prevails through all the theorems which are coming up. We will keep silent about it, but be reassured that it works without any problems.

Lemma 2. Let $T$ be a normal operator on a finite dimensional inner product space or Hermitian inner product space. Then $T$ and $T^*$ are simultaneously diagonalizable.

Proof: If $T$ is normal, then $T^*$ is also normal since $(T^*)^* = T$ and $TT^* = T^*T$. By the previous theorem, both are diagonalizable. But they also commute, so they are simultaneously diagonalizable. □

We can actually upgrade the result above and show that for a normal operator $T$ as above, every eigenvector for $T$ is also an eigenvector for $T^*$ (recall that this is not necessarily true for arbitrary simultaneously diagonalizable operators).

Lemma 3. Let $T$ be a normal operator on a finite dimensional inner product space or Hermitian inner product space $(V, \langle \cdot, \cdot \rangle)$. Then if $v$ is an eigenvector for $T$ with eigenvalue $\lambda$, then $v$ is also an eigenvector for $T^*$, with eigenvalue $\bar{\lambda}$.

Proof: A vector $v \neq 0$ is an eigenvector of $T$ with eigenvalue $\lambda$ if and only if $(T - \lambda I)v = 0$. But this happens iff $\|(T - \lambda I)v\| = 0$. Now, notice that $(T - \lambda I)^* = T^* - \overline{\lambda} I$ and observe that

$$0 = \|(T - \lambda I)v\|^2 = \langle (T - \lambda I)v, (T - \lambda I)v \rangle = \langle v, (T - \lambda I)^* (T - \lambda I)v \rangle = \langle v, (T - \lambda I)(T - \lambda I)^* v \rangle = \langle (T - \lambda I)^* v, (T - \lambda I)^* v \rangle = \|(T - \lambda I)^* v\|^2.$$

This equality proves that $(T^* - \overline{\lambda} I)v = 0$. Hence, $v$ is an eigenvector for $T^*$ with eigenvalue $\overline{\lambda}$. □

Lemma 4. Let $T$ be a normal operator on a finite dimensional inner product space or Hermitian inner product space $(V, \langle \cdot, \cdot \rangle)$. Then different eigenspaces of $T$ are orthogonal to each other. In other words, if $v$ and $w$ are eigenvectors of $T$ having distinct eigenvalues, then $\langle v, w \rangle = 0$. 

Proof: Suppose that \( v \) and \( w \) are non-zero vectors such that \( Tv = \lambda_1 v \) and \( Tw = \lambda_2 w \) with \( \lambda_1 \neq \lambda_2 \). By the previous lemma, \( T^* w = \lambda_2 w \). Then,

\[
\begin{align*}
\lambda_1 \langle v, w \rangle &= \langle \lambda_1 v, w \rangle \\
&= \langle Tv, w \rangle \\
&= \langle v, T^* w \rangle \\
&= \langle v, \lambda_2 w \rangle \\
&= \lambda_2 \langle v, w \rangle.
\end{align*}
\]

This implies that \( (\lambda_1 - \lambda_2) \langle v, w \rangle = 0 \). But since \( \lambda_1 \neq \lambda_2 \), we obtain that \( \langle v, w \rangle = 0 \). \( \square \)

Theorem 2. Let \( T \) be a normal operator on a finite dimensional Hermitian inner product space \((V, \langle, \rangle)\). Then there exists an orthonormal basis \( B \) for \( V \) consisting of eigenvectors of \( T \).

Proof: By the theorem above, we know that \( T \) is diagonalizable. Let us use the notation in the proof of the theorem, so that \( \delta_T(x) = (x - \lambda_1) \ldots (x - \lambda_k) \) is the minimal polynomial of \( T \) where \( \lambda_1, \ldots, \lambda_k \) are the distinct eigenvalues of \( T \). Let \( W_{\lambda_1}, \ldots, W_{\lambda_k} \) be the corresponding eigenspaces. Since \( T \) is diagonalizable, we have

\[
V = W_{\lambda_1} \oplus \ldots \oplus W_{\lambda_k}.
\]

The previous lemma tells us that \( W_{\lambda_i} \perp W_{\lambda_j} \) if \( i \neq j \). Finally, we can choose an orthonormal basis for each \( W_{\lambda_i} \) by Gram-Schmidt orthogonalization, and their union will give us an orthonormal basis \( B \) of eigenvectors of \( T \) for \( V \). \( \square \)

The orthogonal decomposition into eigenspaces in the previous theorem suggests that we can define orthogonal projection operators from \( V \) to these subspaces, and then express everything in terms of these projection operators. This will lead to the spectral theorem, the goal of this lecture.

Recall that a linear operator \( P \) on \( V \) is called a projection if \( P^2 = P \). We also defined orthogonal projections \( \text{proj}_W \) to a subspace \( W \) in inner product spaces. The condition of orthogonality was that

\[
\text{proj}_W(v) \perp (v - \text{proj}_W(v))
\]

for every \( v \in V \). The common abstraction that captures both of these notions is the following:

**Definition 1.** Let \((V, \langle, \rangle)\) be an inner product space or a Hermitian inner product space. A linear operator \( P \) on \( V \) is called an orthogonal projection if

1. \( P^2 = P \),
2. \( P = P^* \), namely \( P \) is self-adjoint.

**Lemma 5.** If \( P \) is an orthogonal projection then \( P v \perp (v - P v) \) for every \( v \in V \).

Proof:

\[
\begin{align*}
\langle P v, v - P v \rangle &= \langle v, P^* (I - P) v \rangle \\
&= \langle v, P (I - P) v \rangle \\
&= \langle v, (P - P^2) v \rangle \\
&= 0.
\end{align*}
\]

\( \square \)
Theorem 3. (Spectral Theorem) Let $T$ be a normal operator on a finite dimensional inner product space or Hermitian inner product space $(V, \langle \cdot, \cdot \rangle)$. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of $T$ and $W_{\lambda_1}, \ldots, W_{\lambda_k}$ be the corresponding eigenspaces. Then there exist orthogonal projections $P_i$ of $V$ such that

1. $\text{Im}(P_i) = W_{\lambda_i}$,
2. $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k$,
3. $P_i + P_2 + \ldots + P_k = I$,
4. $P_i P_j = 0$ for $i \neq j$.

Proof: We already know from above that $V = W_{\lambda_1} \oplus \ldots \oplus W_{\lambda_k}$ and that this is an orthogonal direct sum, namely any two summands are pairwise orthogonal. What is left is to explicitly describe the projection operators. For this we recall the polynomials that came up in Lagrange interpolation long ago in MATH 261. Let

$$p_i(x) = \frac{(x - \lambda_1)(x - \lambda_2)\ldots(x - \lambda_{i-1})(x - \lambda_{i+1})\ldots(x - \lambda_k)}{(\lambda_i - \lambda_1)(\lambda_i - \lambda_2)\ldots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\ldots(\lambda_i - \lambda_k)}.$$ 

Now, set $P_i = p_i(T)$. We claim that $P_i$ satisfy all the conditions above.

First of all, let us compute $P_i$ applied to a vector $v \in V$. By the direct sum decomposition of $V$, the vector can be expressed as

$$v = v_1 + v_2 + \ldots + v_k$$

where $v_i \in W_{\lambda_i}$. Furthermore, such a decomposition for $v$ is unique. It is enough to understand the action of $P_i$ on each of these $v_j$’s. If $j \neq i$, then since $p_i(x)$ is divisible by $x - \lambda_j$ and $(T - \lambda_j I)v_j = (\lambda_j - \lambda_j)v_j = 0$, we see that $P_i v_j = 0$. On the other hand, by direct computation,

$$P_i v_i = \frac{(\lambda_i - \lambda_1)(x - \lambda_2)\ldots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\ldots(\lambda_i - \lambda_k)}{(\lambda_i - \lambda_1)(\lambda_i - \lambda_2)\ldots(\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1})\ldots(\lambda_i - \lambda_k)} v_i = v_i.$$

By using these, we see that

$$P_i(v) = P_i(v_1 + \ldots + v_k) = v_i.$$ 

This immediately shows that $P_i(v) = v_i = P_i^2(v)$, therefore $P_i^2 = P_i$. Furthermore, since $v_i$ is an eigenvalue for $P_i$ with eigenvalue 1, the same is true for $P_i^*$ by lemma and this shows $P_i^* v_i = v_i$. Likewise $P_i^* v_j = 0$ for $j \neq i$. Therefore $P_i = P_i^*$ and consequently $P_i$ is an orthogonal projection.

1. Since $P_i v = v_i$ it is clear that the image of $P_i$ is $W_{\lambda_i}$.

2. Again, by using the fact that $V$ is sum of eigenspaces, it is enough to show that the left hand side and right hand side agree for any eigenvector. If $v_i \in W_{\lambda_i}$, then $Tv_i = \lambda_i v_i$ by definition. Also, $P_i v_i = v_i$ and $P_j v_i = 0$ for $j \neq i$ so the equality holds.

3. In a way similar to (2), it is enough to show that $(P_1 + \ldots + P_k)(v) = v$ for all $v \in V$. Again, since $V$ is a direct sum of eigenspaces, it is enough to show this when $v$ is an eigenvector. So suppose that $v = v_i \in W_{\lambda_i}$. Then $(P_1 + \ldots + P_k)(v_i) = v_i$ so the claim follows.

4. We observe that $\delta_T(x) = (x - \lambda_1)\ldots(x - \lambda_k)$ divides $p_i(x)p_j(x)$ for any $i \neq j$ since the product contains all the linear factors. Therefore $P_i P_j = p_i(T)p_j(T) = 0$. $\square$
1 Consequences of the Spectral Theorem

The spectral theorem for normal operators on a finite dimensional inner product space or Hermitian inner product was proved in the last lecture. This theorem has many consequences for orthogonal, unitary and self-adjoint operators and also for corresponding matrices. We will discuss some of these here.

1.1 Self-Adjoint Operators

Recall that an operator $T$ on an inner product space or a Hermitian inner product space is called self-adjoint if $T^* = T$. All self-adjoint operators are automatically normal, since $T = T^*$. Obviously implies $TT^* = T^*T$.

Proposition 1. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space or Hermitian inner product space and $T$ a normal operator on $V$. Then, $T$ is self-adjoint if and only if all eigenvalues of $T$ are real.

Proof: By the spectral theorem, there exist a set of orthogonal projections $P_1, \ldots, P_k$ such that $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k$. Since $P_i^* = P_i$ for each $i$, we clearly have $T^* = \overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + \ldots + \overline{\lambda_k} P_k$.

Now, if all eigenvalues are real, then $\overline{\lambda_i} = \lambda_i$ for each $i$ and consequently $T = T^*$, so $T$ is self-adjoint. Conversely, if $T = T^*$, we claim that all eigenvalues are real. To see this, apply both $T$ and $T^*$ to an eigenvector $v_i$ of $T$ with eigenvalue $\lambda_i$:

\[ T = T^* \]
\[ Tv_i = T^* v_i \]
\[ (\lambda_1 P_1 + \ldots + \lambda_k P_k)v_i = (\overline{\lambda_1} P_1 + \ldots + \overline{\lambda_k} P_k)v_i \]
\[ \lambda_i P_i v_i = \overline{\lambda_i} P_i v_i \]
\[ \lambda_i v_i = \overline{\lambda_i} v_i \]
\[ \lambda_i = \overline{\lambda_i} \]

In this derivation we used the following facts: $P_j v_i = 0$ if $j \neq i$, $P_i v_i = v_i$, and $v_i \neq 0$ in the last step. This finishes the proof. □

Corollary 1. (a) Let $A$ be a symmetric, real, $n \times n$ matrix. Then all eigenvalues of $A$ are real. There exists a set of $n$ real eigenvectors of $A$ which form an orthogonal basis with respect to the standard inner product on $\mathbb{R}^n$.

(b) Let $A$ be a Hermitian, complex, $n \times n$ matrix. Then all eigenvalues of $A$ are real. There exists a set of $n$ (complex) eigenvectors of $A$ which form an orthogonal basis with respect to the standard Hermitian inner product on $\mathbb{C}^n$.

Proof: (a) Recall that with respect to the standard inner product on $\mathbb{R}^n$, the matrix representing a linear operator is symmetric if and only if the operator is self-adjoint. Now, use the proposition above, and also the result from the previous lecture that for any normal operator one can find
a basis of orthonormal eigenvectors. These eigenvectors must be real since both the matrix $A$ and the eigenvalues are real.

(b) Almost identical to the proof of part (a), except that the eigenvectors need not be real since $A$ is not necessarily real. □

**Corollary 2.** (a) Let $A$ be a symmetric, real $n \times n$ matrix. Then, there exists an **orthogonal** matrix $P$ such that $A = PDP^T$, where $D$ is a diagonal matrix.

(b) Let $A$ be a Hermitian, complex $n \times n$ matrix. Then, there exists a **unitary** matrix $P$ such that $A = PDP^*$ where $D$ is a diagonal matrix.

**Proof:** (a) By part (a) of the corollary above, there exists an orthogonal basis of real eigenvectors for $A$. Divide each of these by its norm to make the basis orthonormal. Let $P$ be the matrix whose columns are these $n$ eigenvectors. Then, the orthonormal condition immediately implies that $P^TP = I$. But then also $P^TP = I$ and $P$ is an orthogonal matrix. By the change of basis theorem, $A = PDP^{-1} = PDP^T$ where $D$ is the diagonal matrix whose diagonal entries are the eigenvalues of $A$.

(b) This is similar to part (a) again. By part (b) of the corollary above, there exists an orthogonal basis of possibly complex eigenvectors for $A$, with respect to the standard Hermitian inner product. Let $P$ be the matrix whose columns are these $n$ eigenvectors. Then, one has $P^*P = I$, hence also $PP^* = I$. Therefore, $P$ is unitary. Again, $A = PDP^{-1} = PDP^*$ where $D$ is the diagonal matrix of eigenvalues of $A$. □

**Remark:** Part (a) actually follows from part (b) since any unitary matrix which is real is orthogonal. The two are stated separately just for emphasis and ease of use.

**Example:** Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

without making any detailed computations. Write $A = PDP^T$ where $P$ is an orthogonal matrix and $D$ a diagonal matrix.

**Solution:** The matrix is symmetric, so by above, it must have real eigenvalues and an orthogonal set of eigenvectors. Since the sum of the first and third rows is equal to twice the second row, the matrix is singular, hence $\lambda_1 = 0$ is an eigenvalue. The vector

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

is a 0-eigenvector by observation. Also observe that the sum of each row of $A$ is equal to 3. Therefore

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

must an eigenvector with eigenvalue $\lambda_2 = 3$. To find the third eigenvalue, notice that $Tr(A) = 5$ and $Tr(A)$ must be equal to the sum of the three eigenvalues. So $\lambda_3 = 2$. All eigenspaces are necessarily 1 dimensional. Hence to find the third eigenvector, just pick anything orthogonal to
both $v_1$ and $v_2$, for instance,

$$v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Finally, in order to form $P$, divide each of $v_1, v_2, v_3$ by their norms and put these into the columns of $P$:

$$P = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

On the other hand, $D$ is just the diagonal matrix with eigenvalues as diagonal entries. Namely,

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

### 1.2 Orthogonal and Unitary Operators

Since an operator $T$ on an inner product space is orthogonal if and only if it is unitary and its matrix representation with respect to the standard basis has real entries, it is reasonable to prove some structural results for unitary operators first. Properties of orthogonal operators will follow immediately. Recall that $T$ is unitary if

$$TT^* = TT^* = I.$$  

**Proposition 2.** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional Hermitian inner product space and $T$ a normal operator on $V$. Then, $T$ is unitary if and only if all eigenvalues of $T$ have modulus equal to 1 (i.e. they lie on the unit circle, or equivalently, each eigenvalue is of the form $e^{i\theta}$ for some $\theta$).

**Proof:** By the spectral theorem, there exist orthogonal projections $P_1, P_2, \ldots, P_k$ such that $T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k$ and $P_1 + P_2 + \ldots + P_k = I$. Now, we know that $T^* = \overline{\lambda} P_1 + \overline{\lambda_2} P_2 + \ldots + \overline{\lambda_k} P_k$. Therefore,

$$TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k)(\overline{\lambda_1} P_1 + \overline{\lambda_2} P_2 + \ldots + \overline{\lambda_k} P_k)$$

$$= \lambda_1 \overline{\lambda_1} P_1^2 + \lambda_2 \overline{\lambda_2} P_2^2 + \ldots + \lambda_k \overline{\lambda_k} P_k^2$$

$$= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \ldots + |\lambda_k|^2 P_k$$

Here, we used the fact that $P_i P_j = 0$ if $i \neq j$ and that $P_i^2 = P_i$. Now, if each eigenvalue has modulus 1, then $|\lambda_i|^2 = 1$ for all $i$ and it is clear that $TT^* = I$, so $T$ is unitary. Conversely, suppose that $T$ is unitary, so that $TT^* = I$. Then suppose that $v_i$ is an eigenvector of $T$ with eigenvalue $\lambda_i$. Then,

$$TT^* = I$$

$$|\lambda_1|^2 v_1 + |\lambda_2|^2 v_2 + \ldots + |\lambda_k|^2 v_k = P_1 v_1 + P_2 v_2 + \ldots + P_k v_k$$

$$|\lambda_i|^2 v_i = P_i v_i$$

$$|\lambda_i|^2 = v_i$$

$$|\lambda_i|^2 = 1$$
1 CONSEQUENCES OF THE SPECTRAL THEOREM

Again, $P_j v_i = 0$ for $j \neq i$ and $P_i v_i = v_i$ were used here, along with $v_i \neq 0$ at the last step. This proves that all eigenvalues of a unitary operator must lie on the unit circle, hence the proof is finished. □

As an immediate consequence, we get

**Corollary 3.** Let $T$ be a normal linear operator on a finite dimensional real inner product space. Then $T$ is an orthogonal operator if and only if all eigenvalues of $T$ lie on the unit circle.

**Corollary 4.** Let $A$ be an orthogonal or unitary matrix. Then the eigenvalues of $A$ lie on the unit circle. Furthermore, there exists a unitary matrix $P$ and a diagonal (complex) matrix $D$ such that $A = PDP^*$.

**Proof:** $A$ represents an orthogonal or unitary operator on $\mathbb{R}^n$ together with its standard inner product or on $\mathbb{C}^n$ with its standard Hermitian inner product respectively. Therefore, by above, its eigenvalues are on the unit circle. Furthermore, since $A$ is normal, it has an orthogonal basis of eigenvectors. Let $P$ be the matrix with these basis elements as its columns. Then $PP^* = P^*P = I$ so $P$ is a unitary matrix. Let $D$ be the (complex) diagonal matrix whose entries are eigenvalues of $A$. Then $A = PDP^{-1} = PDP^*$. □

**Caution:** Even when $A$ is a orthogonal matrix, hence has real entries, the eigenvalues and eigenvectors of $A$ will usually be complex.

**Corollary 5.** Let $T$ be an orthogonal operator on $\mathbb{R}^n$ with respect to its standard inner product. Then there exists an orthonormal basis of $\mathbb{R}^n$ such that the matrix $A$ representing $T$ in this basis is in the block-diagonal form

$$a = \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_k \end{bmatrix}$$

where each $A_j$ is one of the following $1 \times 1$ or $2 \times 2$ matrices for certain values of $\theta_j$:

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{bmatrix}$$

**Proof:** We know that $T$ is diagonalizable, but with complex eigenvalues and eigenvectors. Since $T$ is real (its representation with respect to any basis of $\mathbb{R}^n$ has real entries), if $v$ is an eigenvector with eigenvalue $\lambda$, then

$$Tv = \lambda v$$
$$\overline{Tv} = \overline{\lambda v}$$
$$T\overline{v} = \overline{\lambda \overline{v}}$$

hence $\overline{v}$ is also an eigenvector of $T$, with eigenvalue $\overline{\lambda}$.

Now, let us think about the different possibilities: If an eigenvalue $\lambda$ is real, then $v$ will also be real. Since $T$ is orthogonal its eigenvalues lie on the unit circle, so if $\lambda \in \mathbb{R}$ and $\lambda$ is on the unit circle then $\lambda = \pm 1$. This takes care of the real eigenvalues and the corresponding $1 \times 1$ matrices in the block-diagonal representation.
If $\lambda$ is not real, we still know that it is on the unit circle and we can write $\lambda = e^{i\theta_j}$ for some $\theta_j$. If $v$ is a corresponding eigenvector, write $v = a + ib$ where $a$ and $b$ are the real and imaginary parts of the vector $v$. By explicit computation,

\[
Tv = \lambda v = e^{i\theta_j}(a + ib) = (\cos \theta_j + i \sin \theta_j)(a + ib) = (\cos \theta_j a - \sin \theta_j b) + i(\sin \theta_j a + \cos \theta_j b)
\]

On the other hand, $T$ is real, so $Ta$ must be equal to the real part of $Tv$ and $Tb$ must be equal to the imaginary part of $Tv$. This gives us

\[
Ta = \cos \theta_j a - \sin \theta_j b \\
Tb = \sin \theta_j a + \cos \theta_j b
\]

Now, let $W_j = \text{Span}(a, b)$. This is a 2 dimensional subspace which is clearly $T$-invariant. Furthermore, since $T$ rotates both $a$ and $b$ inside $W_j$ by $\theta_j$ radians, we see that $T$ restricted to $W_j$ is a rotation by $\theta_j$ radians. We can then select an orthogonal basis $\{u_j, w_j\}$ for $W_j$ and put these in our basis. By iterating this procedure for each eigenvalue, one can complete the proof.

A linear operator on $\mathbb{R}^3$ is called a rotation centered at the origin if it is orthogonal with respect to the standard inner product and has determinant 1.

Corollary 6. If $T$ is a rotation of $\mathbb{R}^3$ centered at the origin, then it has an axis. In other words, it has an eigenvector $v$ with $Tv = v$.

Proof: It suffices to show that $+1$ is an eigenvalue of $T$. We will use the result above. If all three eigenvalues of $T$ are real, then each of them is $\pm 1$. But $T$ has determinant $+1$, so the product of the eigenvalues is $+1$. So at least one of them must be $+1$. If $T$ has a non-real eigenvalue, then by above, there exists a basis of $\mathbb{R}^3$ and a value of $\theta$ such that the matrix of $T$ with respect to this basis is

\[
A = \begin{bmatrix}
\pm 1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{bmatrix}
\]

But $\det(A) = 1$, hence the top left entry must be $+1$. This finishes the proof. \qed
1 Quadratic Forms

1.1 Definition

Let $F$ be a field of characteristic different from 2 for the following discussion. We want to look at quadratic forms on $F$, namely functions given by degree 2 homogeneous polynomials in several variables. More formally,

**Definition 1.** Let $F$ be a field of characteristic different from 2 and $n \geq 1$ an integer. A quadratic form in $n$ variables $x_1, \ldots, x_n$ is degree 2 homogeneous polynomial $q$ of the form

$$q(x_1, x_2, \ldots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j$$

for some $a_{ij} \in F$.

**Example:** Take $F = \mathbb{R}$ and $n = 2$. A real quadratic form in 2 variables $x, y$ is of the form

$$q(x, y) = ax^2 + bxy + cy^2$$

for some $a, b, c \in \mathbb{R}$.

**Remark 1:** Compared to the usual usage of a quadratic function or quadratic polynomial in calculus, analytic geometry or other places, notice that the quadratic forms here are homogeneous, namely all summands have degree exactly 2 (and not lower degree).

**Remark 2:** A quadratic form immediately defines a function $q : F^n \to F$. But we still wish to keep the distinction between polynomials and functions here as well, since in principle different polynomials could give us the same function (though it won’t happen in the current case since $\text{char}(F) \neq 2$).

1.2 Representing a Quadratic Form by a Symmetric Matrix

Suppose that $B = (b_{ij})$ is an $n \times n$ matrix in $M_{n \times n}(F)$ where $F$ is a field of characteristic different from 2. Let $\mathbf{x}$ denote the $n \times 1$ column vector of variables:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Then the matrix $B$ defines a quadratic form by the formula

$$q(\mathbf{x}) = \mathbf{x}^T B \mathbf{x}.$$
We can write a formula for this quadratic form explicitly in terms of the entries of $B$ simply by carrying out the matrix multiplication:

$$q(x) = x^T B x$$

$$= \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \sum_i b_{ii} x_i^2 + \sum_{i<j} (b_{ij} + b_{ji}) x_i x_j$$

We can replace $B$ by a symmetric matrix $A = (a_{ij})$ in order to make the correspondence between the matrices and quadratic forms a bijection. Simply set

$$a_{ij} = \frac{b_{ij} + b_{ji}}{2}$$

(notice that 2 is invertible since $\text{char}(F) \neq 2$). It is clear that $A$ gives us the same quadratic form as $B$ and that $A$ is symmetric. Furthermore, it is a simple exercise to show that this correspondence between $n \times n$ symmetric matrices and quadratic forms in $n$ variables is a bijection.

**Example:** Let $q$ be the complex quadratic form defined in 2 variables $x, y$ by the formula

$$q(x, y) = (2 + i)x^2 + 4ixy - 3y^2.$$ 

Then $q$ can be written in the matrix form described above as

$$q(x, y) = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2 + i & 2i \\ 2i & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

### 1.3 Equivalence of Quadratic Forms

Let $F$ be a field with characteristic different from 2. We now want to describe an equivalence relation on quadratic forms of $n$ variables over $F$. By above, this is going to give us an equivalence relation on the set of $n \times n$ symmetric matrices over $F$. Essentially, we want to say that two quadratic forms are equivalent if the vector of variables is changed by an invertible linear change of variables.

**Definition 2.** Suppose that $q_1(x) = x^T A_1 x$ and $q_2(y) = y^T A_2 y$ be two quadratic forms in $n$ variables where $A_1, A_2$ are symmetric matrices in $M_{n \times n}(F)$ where $\text{char}(F) \neq 2$. We say that $q_1$ is *equivalent* to $q_2$ if there exists an invertible matrix $P$ in $M_{n \times n}(F)$ such that

$$A_2 = P^T A_1 P.$$ 

**Remark:** The definition arises from the following observation. Suppose we make a linear, invertible change of variables

$$x = Py.$$
Then, write the form \( q_1(x) \) in terms of the new variable (vector) \( y \):

\[
q_1(x) = x^T A_1 x = (Py)^T A_1 (Py) = y^T (P^T A_1 P) y
\]

So, \( A_2 \) in the definition is just the matrix in the middle of this expression when \( q_1 \) is written in terms of \( y \); in this sense the definition is quite natural.

**Proposition 1.** Equivalence of quadratic forms is an equivalence relation.

**Proof:** Denote the relation by the notation \( q_1 \sim q_2 \) throughout.

(i) Reflexivity: Since \( P = I \) is invertible and \( I^T A I = A \) for any \( A \), it is clear that \( q \sim q \) for any \( q \). Therefore, the relation is reflexive.

(ii) Symmetry: Suppose that \( q_1 \sim q_2 \), namely \( A_2 = P^T A_1 P \) for some invertible matrix \( P \). But now, using the fact that \( (P^T)^{-1} = (P^{-1})^T \), we get

\[
A_2 = P^T A_1 P, \quad (P^T)^{-1} A_2 P^{-1} = A_1, \quad (P^{-1})^T A_2 P^{-1} = A_1
\]

Therefore, \( q_2 \sim q_1 \) via the matrix \( P^{-1} \). Therefore, the relation is symmetric.

(iii) Transitivity: Suppose that \( q_1 \sim q_2 \) and \( q_2 \sim q_3 \) where these quadratic forms are given by symmetric matrices \( A_1, A_2 \) and \( A_3 \) respectively. Then there exist invertible \( P, Q \) such that,

\[
A_2 = P^T A_1 P, \quad A_3 = Q^T A_2 Q.
\]

Now, we can combine the two relations to get

\[
A_3 = Q^T A_2 Q = Q^T P^T A_1 P Q = (PQ)^T A_1 (PQ).
\]

Noting that \( PQ \) is an invertible matrix, we see from this equality that \( q_1 \sim q_3 \). Hence, the relation is transitive. This finishes the proof. \( \Box \)

### 1.4 Diagonalization of Quadratic Forms

**Definition 3.** A quadratic form \( q \) in \( n \) variables is **diagonal** if there exist \( d_1, \ldots, d_n \in F \) such that

\[
q(x) = \sum d_i x_i^2.
\]

In other words, \( q \) is diagonal if the symmetric matrix representing it is a diagonal matrix.

**Theorem 1.** (Diagonalization of quadratic forms) Let \( F \) be a field with \( \text{char}(F) \neq 2 \) and \( q \) a quadratic form in \( n \) variables over \( F \). Then \( q \) is equivalent to a diagonal quadratic form.
Proof: Suppose that \( q(x) = x^T A x \) where \( A \) is a symmetric \( n \times n \) matrix over \( F \). It will be enough to show that there exists an invertible \( n \times n \) matrix \( P \), so that

\[
D = P^T A P
\]
is a diagonal matrix. We do this by induction on \( n \). For \( n = 1 \), \( A \) is already diagonal, so the assertion clearly holds. Suppose it holds up to \( n - 1 \) and let us prove it for \( n \). Say \( A = (a_{ij}) \).

Recall that, multiplying \( A \) on the left by an elementary matrix corresponds to an elementary row operation, and multiplying \( A \) on the right by the transpose of the elementary matrix corresponds to the same operation applied to the columns of \( A \).

First, suppose that \( a_{11} \neq 0 \). For \( i = 2, \ldots, n \) say \( E_i \) is the elementary matrix corresponding to the following row operation of the 3rd type:

\[
-\frac{a_{1i}}{a_{11}} R_1 + R_i \rightarrow R_i.
\]

Upon multiplying \( A \) on the left by \( Q = E_n E_{n-1} \ldots E_2 \), all elements in column 1 except the 1,1 entry will be cleared. Now, multiplying on the right by \( Q^T = E_T^n \ldots E_2^T \) will also clear all elements in row 1 except the 1,1 entry, because \( A \) was symmetric to start with. Therefore,

\[
Q A Q^T = \begin{bmatrix} a_{11} & 0 \\ 0 & B \end{bmatrix}
\]

(in block-diagonal form) where \( B \) is an \( (n - 1) \times (n - 1) \) matrix. The matrix \( B \) is symmetric since \( (Q A Q^T)^T = Q A^T Q^T = Q A Q^T \). Therefore, by the inductive assumption, there exists an invertible \( (n - 1) \times (n - 1) \) matrix \( P' \) such that \( (P')^T B P' \) is diagonal. Setting

\[
P = Q^T \begin{bmatrix} 1 & 0 \\ 0 & P' \end{bmatrix}
\]

we see that \( P^T A P \) must also be diagonal.

It remains to settle the case where \( a_{11} = 0 \). If \( a_{1i} = 0 \) also for all \( i = 2, \ldots, n \), then the whole first column, and by symmetry the first row, is 0, so we are done by induction on the lower right corner \( (n - 1) \times (n - 1) \) submatrix of \( A \). If \( a_{1i} \neq 0 \) for some \( i \), then let \( E \) be the elementary matrix corresponding to the row operation

\[
R_i + R_1 \rightarrow R_1.
\]

Then \( E A E^T \) will have non-zero 1,1 term (we need \( \text{char}(F) \neq 2 \) right here as well! Why?), so we can proceed and finish the proof as before. This settles the induction step and hence finishes the proof. \( \square \).

Remark: A diagonal quadratic form equivalent to a given quadratic form is almost never unique. The proof above gives us just one way to find such a diagonal quadratic form.

Exercise: Over \( F = \mathbb{R} \), find diagonal quadratic forms equivalent to

(a) \( q_1(x, y) = x^2 + 4xy + y^2 \),
(b) \( q_2(x, y) = x^2 + 2xy + y^2 \),
(c) \( q_3(x, y) = x^2 + xy + y^2 \).
In the previous lecture, we defined quadratic forms, their equivalence, and showed that any quadratic form over a field of characteristic different from 2 is equivalent to a diagonal quadratic form. In this lecture, we want to specialize to $F = \mathbb{R}$ and obtain stronger results in this particular but important case.

1 Principal Axis Theorem

Let $F = \mathbb{R}$ be the field of real numbers. Consider a quadratic form $q(x)$ over $\mathbb{R}$ defined by

$$q(x) = x^T A x$$

where $A$ is symmetric, real $n \times n$ matrix. By the results of the previous lecture, we know that there exists an invertible matrix $P$ such that $P^T A P$ is a diagonal matrix. But the matrix $P$ is far from being unique. In the real case, we can make the following special choice for $P$ which has geometric significance:

**Theorem 1.** (Principal Axis Theorem) Suppose that $q(x) = x^T A x$ is a quadratic form in $n$ variables. Regard $\mathbb{R}^n$ as an inner product space with its usual standard inner product. Then there exists an orthogonal matrix $P$ such that $P^T A P$ is diagonal (i.e. the quadratic form $q$ can be diagonalized by an orthogonal change of basis).

**Proof:** Since $A$ is a symmetric matrix, it represents a self-adjoint operator on $\mathbb{R}^n$ with its standard inner product. By the spectral theorem, $A$ has real eigenvalues and it can be unitarily diagonalized. Namely, there exists an orthogonal matrix $P$ such that $P^T A P$ is diagonal. This finish the proof. □

This proof looks misleadingly short, but beware that it uses a powerful tool, namely the spectral theorem, with all its might.

**Example:** Suppose that $q(x, y)$ is the following quadratic form in 2 variables:

$$q(x, y) = x^2 + xy + y^2.$$ 

Let us find an orthogonal diagonalization of this form as in the principal axis theorem. First of all, write $q$ in matrix form:

$$q(x, y) = [x\ y] \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^T A x.$$

We now want to find an orthogonal matrix $P$ such that $P^T A P$ is diagonal. This is the same problem as finding an orthonormal basis of eigenvectors for $A$. The characteristic polynomial of $A$ is

$$\Delta_A(x) = \det(xI - A) = (x - 1)^2 - \frac{1}{4} = \left( x - \frac{3}{2} \right) \left( x - \frac{1}{2} \right).$$
Therefore the eigenvalues of $A$ are $\lambda_1 = \frac{3}{2}$ and $\lambda_2 = \frac{1}{2}$. We can compute their eigenspaces and obtain

\[ W_{\lambda_1} = \text{Span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right), \quad W_{\lambda_2} = \text{Span} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \]

Of course, we have $\mathbb{R}^2 = W_{\lambda_1} \oplus W_{\lambda_2}$ and $W_{\lambda_1} \perp W_{\lambda_2}$ as expected. To get the columns of $P$, just divide the eigenvectors by their norms. We get

\[ P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad D = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \]

Then we will have $P^TAP = D$ as desired. What precisely happened to the quadratic form is the following: If we make a change of variables

\[ \begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} z + w \\ \frac{z - w}{\sqrt{2}} \end{bmatrix} \]

then in terms of the new set of variables $z, w$ our quadratic form becomes

\[ q(z, w) = \frac{3}{2} z^2 + \frac{1}{2} w^2. \]

This can of course be checked also by direct substitution in the original quadratic form.

1.1 Quadratic Forms in $n$ Variables and Quadratic Equations in $n - 1$ Variables

Notice that a quadratic form $q(x)$ in $n$ variables, which by definition is homogeneous, determines a certain zero locus $C$ in $\mathbb{R}^n$: $C$ is just the set of all points $x$ in $\mathbb{R}^n$ for which the quadratic form $q$ takes the value 0. The set $C$ is a cone, since by the homogeneity of $q$, for any $c \in \mathbb{R}$ we have

\[ q(cx) = c^2 q(x) \]

and this implies that if $x \in C$ then $cx \in C$ for all $c \in \mathbb{R}$. Namely, any line through a point on $C$ and the origin totally lies on $C$. In this respect, $C$ is a cone (this actually is the definition of a cone).

One may also be interested in the zero locus of a non-homogeneous quadratic polynomial, in other words a polynomial with degree 0 and degree 1 terms as well as degree 2 terms. One simple way to obtain such a quadratic polynomial $f$ in $n - 1$ variables from a quadratic form $q$ in $n$ variables is to set one of the $n$ variables to be equal to 1. For the sake of a concrete example, say our quadratic form is

\[ q(x, y, z) = x^2 + 2xy + 3y^2 + xz + 5yz + 7z^2. \]

Then the corresponding quadratic polynomial will be

\[ f(x, y, z) = x^2 + 2xy + 3y^2 + x + 5y + 7. \]

(occasionally, this procedure might create a lower degree polynomial, but this will not be our concern for now.) It is also easy to see how this process can be reversed.
Geometrically, what is happening is that we are intersecting the cone $C$ with an affine hyperplane $x_n = 1$ in order to get the zero locus $Q$ of some quadratic polynomial. In analytic geometry, one often is interested in understanding such $Q$. Then one can pass to the cone $C$ and the quadratic form $q$, use the principal axis theorem, and obtain consequences for $Q$ itself.

**Example:** Find the principal axes of the ellipse

$$2x^2 + 6xy + 10y^2 = 4.$$ How many degrees should this ellipse be rotated so that one of its axes is aligned with the $x$-axis?

To solve this problem, let us first reverse the procedure above and get a quadratic form in 3 variables. First set $f(x, y) = 2x^2 + 6xy + 10y^2 - 4$. Then

$$q(x, y, z) = 2x^2 + 6xy + 10y^2 - 4z^2.$$ Let us now write $q$ by using a symmetric matrix:

$$q(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x^TAx.$$ Next, let us find the eigenvalues and an orthonormal eigenbasis for $A$.

$$\Delta_A(x) = ((x - 2)(x - 10) - 9)(x + 4) = x^2 - 12x + 11(x + 4) = (x - 1)(x - 11)(x + 4)$$ therefore the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 11$ and $\lambda_3 = -4$. The eigenspaces can be computed in the standard way:

$$W_{\lambda_1} = \text{Span} \left(\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}\right), \quad W_{\lambda_2} = \text{Span} \left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\right), \quad W_{\lambda_3} = \text{Span} \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

Since $A$ is symmetric, the eigenspaces are orthogonal to each other and their direct sum is $\mathbb{R}^3$. Now, our orthogonal matrix for diagonalization and the diagonal matrix can be taken to be

$$P = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

Then we will have $P^TAP = D$. Setting

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = P \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

we get the diagonalized form to be

$$q(x', y', z') = (x')^2 + 11(y')^2 - 4(z')^2$$ Notice that $z' = z$ so nothing changed in that direction. The operator $P$ is a rotation with $z$-axis as the rotation axis, so this gives us a rotation in the $xy$-plane, given by the upper left $2 \times 2$ corner of $P$. To find the angle of rotation, we must find $\theta$ so that

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{\sqrt{10}}{2} & \frac{\sqrt{10}}{2} \end{bmatrix}$$

which gives approximately $\theta = -18.4^\circ$. 

2 Sylvester’s Law of Inertia

Let $F = \mathbb{R}$ again, and let $q(x)$ be a quadratic form in $n$ variables, given by a symmetric $n \times n$ matrix $A$, so that

$$q(x) = x^T A x.$$ 

We know that the quadratic form $q$ can be diagonalized, in other words there exists an invertible matrix $P$ such that

$$P^T A P = D$$

is a diagonal matrix.

The matrix $P$ above is far from being unique, and as opposed to the previous section, we will not insist that this is an orthogonal diagonalization, namely $P$ need not be orthogonal. The question is, even though $P$ is not unique and therefore $D$ is not unique (caution: its diagonal entries need not be eigenvalues in the general case), are there some features of $D$ which are not changed when we change $P$? In more fancy language, what are the invariants of real quadratic forms under equivalence?

The first and the easier invariant is rank, namely the number $\text{rank}(A)$. Indeed, $P$ is an invertible matrix, therefore

$$\text{rank}(A) = \text{rank}(P^T A P) = \text{rank}(D)$$

therefore the rank of a quadratic form is obviously an invariant under equivalence of quadratic forms. For example, the quadratic forms $q_1(x, y) = x^2 + 2xy + y^2$ has rank 1 and $q_2(x, y) = x^2 + 4xy + y^2$ has rank 2 (check!), hence $q_1$ and $q_2$ cannot be equivalent.

The second and more subtle invariant is signature. This is the content of the following theorem:

**Theorem 2. (Sylvester’s Law of Inertia)** Suppose that $A$ is an $n \times n$ real symmetric matrix. Suppose that $P_1, P_2$ are invertible matrices such that $P_1^T A P_1 = D_1$ and $P_2^T A P_2 = D_2$ are both diagonal matrices. Then the number of positive diagonal entries of $D_1$ is equal to the number of positive diagonal entries of $D_2$. Similarly the number of negative diagonal entries is also the same for $D_1$ and $D_2$.

**Proof:** Since $\text{rank}(D_1) = \text{rank}(D_2)$ as we observed above and for a diagonal matrix rank is equal to the number of non-zero diagonal entries, it is enough to show that the number of positive diagonal entries match. Suppose to the contrary that $D_1$ has $r$ positive diagonal entries, $D_2$ has $s$ positive diagonal entries and $r > s$ without loss of generality. We have

$$A = (P_1^T)^{-1} D_1 P_1^{-1} = (P_2^T)^{-1} D_2 P_2^{-1} \Rightarrow D_1 = P_1^T ((P_2^T)^{-1} D_2 P_2^{-1}) P_1$$

hence if we set $P = P_2^{-1} P_1$ then we will have $D_1 = P^T D_2 P$. Suppose that the first $r$ diagonal entries $d_1, \ldots, d_r$ of $D_1$ are positive, again without loss of generality.

Now, say

$$x = \begin{bmatrix} x_1 \\
 x_2 \\
 \vdots \\
 x_r \\
 0 \\
 \vdots \\
 0 \end{bmatrix}$$


is a vector with the last \( n - r \) entries equal to 0. Then since the first \( r \) diagonal entries of \( D_1 \) are positive, we must have

\[
x^T D_1 x = d_1 x_1^2 + d_2 x_2^2 + \ldots + d_r x_r^2 \geq 0
\]

with equality only in the case \( x = 0 \).

On the other hand, say \( y = Px \).

Suppose that the positive entries of \( D_2 \) are at positions \( i_1, i_2, \ldots, i_s \). Since \( s < r \), the \( s \times r \) homogenous linear system

\[
\begin{align*}
y_{i_1} &= p_{i_11} x_1 + \ldots + p_{i_1r} x_r = 0 \\
y_{i_2} &= p_{i_21} x_1 + \ldots + p_{i_2r} x_r = 0 \\
&\vdots \\
y_{i_s} &= p_{i_s1} x_1 + \ldots + p_{i_s r} x_r = 0
\end{align*}
\]

has a nontrivial solution \( x \), where \( P = (p_{ij}) \). But now, we have the following contradiction:

\[0 \geq y^T D_2 y = x^T D_1 x > 0.\]

The first inequality holds since the \( y_i \) at all positions where \( D_2 \) has a positive entry are set to be 0. This contradiction finishes the proof. \( \Box \)

**Definition 1.** The *signature* of a real quadratic form \( q \) defined by a symmetric matrix \( A \) is the difference between the number of positive eigenvalues of \( A \) and the number of negative eigenvalues of \( A \).

Since by the principal axis theorem each real quadratic form can be diagonalized by an orthogonal change of coordinates in which case the diagonal matrix has eigenvalues of \( A \) as diagonal entries, we see by Sylvester’s law of inertia that for any other non-orthogonal diagonalization as well, the number of positive diagonal entries minus the number of negative diagonal entries is equal to the signature of \( q \). Thus, the signature and rank of \( q \) are invariants under equivalence of quadratic forms.

**Exercise:** Compute the signatures of the quadratic forms \( q_1(x, y) = x^2 + 4xy + y^2 \) and \( q_2(x, y) = x^2 + xy + y^2 \). Conclude that \( q_1 \) and \( q_2 \) are not equivalent.
Let $A$ be any $n \times m$ real or complex matrix. The goal of this lecture is to prove that $A$ admits a singular value decomposition, namely:

- In the real case, there exists an $n \times n$ orthogonal matrix $U$, an $m \times m$ orthogonal matrix $V$ and an $n \times m$ diagonal matrix $\Sigma$ with nonnegative entries, such that $A = U\Sigma V^T$,

- In the complex case, there exists an $n \times n$ unitary matrix $U$, an $m \times m$ unitary matrix $V$ and an $n \times m$ diagonal matrix $\Sigma$ with nonnegative real entries, such that $A = U\Sigma V^*$

A few remarks are in order. First of all, the theorem is valid for matrices of any size (not necessarily square), so this is a rather strong result. Second, for an $n \times m$ matrix $\Sigma$, being diagonal again means that $\Sigma_{ij} = 0$ if $i \neq j$. In the special case that $A$ is a square matrix which is normal, we know that $A$ has a unitary diagonalization, namely $A = UDU^*$ for a diagonal matrix $D$ and a unitary matrix $U$. Therefore the singular value decomposition is one way to generalize unitary diagonalization of normal matrices to all matrices.

It is also worthy of mention that even if a square matrix $A$ is not diagonalizable it still has a singular value decomposition. Therefore, the diagonal entries of $\Sigma$ should not be expected to be the eigenvalues of $A$ in general. They are rather called the singular values of $A$.

1 Positive definite and non-negative definite matrices

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space or a Hermitian inner product space. Recall from previous lectures that an operator $T$ on $V$ is called positive if $T = T^*$ and $\langle Tv, v \rangle > 0$ for all non-zero vectors $v \in V$. Likewise, $T$ is called non-negative if $T = T^*$ and $\langle Tv, v \rangle \geq 0$ for all $v \in V$.

**Lemma 1.** Let $T$ be a self-adjoint operator on a finite dimensional inner product space or a Hermitian inner product space. Then

1. $T$ is non-negative if and only if all its eigenvalues are non-negative,
2. $T$ is positive if and only if all its eigenvalues are positive.

**Proof:** Since $T$ is self-adjoint, it is unitarily diagonalizable with real eigenvalues. Suppose that $v$ is an eigenvector of $T$ with eigenvalue $\lambda$. Then,

$$\langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda ||v||^2.$$  

Since $v \neq 0$, we have $||v||^2 > 0$. Now, if $T$ is a non-negative operator then $\langle Tv, v \rangle \geq 0$ so clearly $\lambda \geq 0$. If $T$ is a positive operator, then $\langle Tv, v \rangle > 0$ since $v \neq 0$, therefore $\lambda > 0$. This proves one direction. Conversely, suppose that all eigenvalues of $T$ are $\geq 0$ (respectively, $> 0$). Write the spectral decomposition of $T$:  

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_k P_k.$$  

where $P_1 + \ldots + P_k = I$. Take any non-zero vector $v \in V$. Decompose it as  

$$v = P_1 v + P_2 v + \ldots + P_k v = v_1 + \ldots + v_k.$$
Then, \( v_i \) is an eigenvector of \( T \) with eigenvalue \( \lambda_i \) and \( v_i \) is orthogonal to \( v_j \) if \( i \neq j \). We have

\[
\langle T v, v \rangle = \langle \lambda_1 v_1 + \ldots + \lambda_k v_k, v_1 + \ldots + v_k \rangle \\
= \lambda_1 ||v_1||^2 + \ldots + \lambda_k ||v_k||^2 \\
\geq 0
\]

So \( T \) is a non-negative operator. In the case that all \( \lambda_i > 0 \), then equality to 0 at the last step above can happen only if all \( v_i = 0 \), namely only if \( v = 0 \). □

Taking \( V \) to be \( \mathbb{R}^n \) with its standard inner product or \( \mathbb{C}^n \) with its standard Hermitian inner product, we have the following analogous definitions for matrices.

**Definition 1.**

1. Let \( A \) be an \( n \times n \) real, symmetric matrix. Then \( A \) is called **non-negative definite** if \( v^T A v \geq 0 \) for all \( n \times 1 \) column vectors \( v \in \mathbb{R}^n \). It is called **positive definite** if \( v^T A v > 0 \) for all \( v \neq 0 \).

2. Let \( A \) be an \( n \times n \) complex, Hermitian matrix. Then \( A \) is called **non-negative definite** if \( v^* A v \geq 0 \) for all \( n \times 1 \) column vectors \( v \in \mathbb{C}^n \). It is called **positive definite** if \( v^* A v > 0 \) for all \( v \neq 0 \).

**Corollary 1.** An \( n \times n \) symmetric real matrix (or Hermitian complex matrix) \( A \) is non-negative definite if and only if all of its eigenvalues are non-negative. It is positive definite if and only if all of its eigenvalues are positive.

**Proof:** This is an immediate consequence of the lemma above. □

We will need one additional result for what follows.

**Proposition 1.**

1. Let \( A \) be any \( n \times m \) matrix. Then the matrix \( A^* A \) and \( AA^* \) are both non-negative definite. (These agree with \( A^T A \) and \( AA^T \) in the real case.)

2. Let \( A \) be any \( n \times m \) matrix of rank \( m \) (therefore \( m \leq n \)). Then the matrix \( A^* A \) is positive definite.

**Proof:** Let us first remark that \( AA^* \) and \( A^* A \) are both self-adjoint, so they are unitarily diagonalizable with real eigenvalues.

1. It is enough to prove that all eigenvalues of \( A^* A \) and \( AA^* \) are non-negative. Say \( v \) is an eigenvector of \( A^* A \) with eigenvalue \( \lambda \). Then

\[
v^* A^* A v = \lambda v^* v = \lambda ||v||^2.
\]

On the other hand,

\[
v^* A^* A v = (Av)^*(Av) = ||Av||^2 \geq 0.
\]

From these two equations, it is clear that \( \lambda = \frac{||Av||^2}{||v||^2} \geq 0 \).

2. Say \( A \) has rank \( m \). Then by the rank-nullity theorem, its kernel is trivial. So if \( v \) is a non-zero vector, then \( Av \neq 0 \). But then, by the first part \( \lambda = \frac{||Av||^2}{||v||^2} > 0 \). □
2 Singular Value Decomposition

Theorem 1. (Singular Value Decomposition)

1. Suppose that $A$ is an $n \times m$ real matrix. Then, there exists an $n \times n$ orthogonal matrix $U$, an $m \times m$ orthogonal matrix $V$, and an $n \times m$ diagonal matrix $\Sigma$ with non-negative diagonal entries such that $A = U\Sigma V^T$.

2. Suppose that $A$ is an $n \times m$ complex matrix. Then, there exists an $n \times n$ unitary matrix $U$, an $m \times m$ unitary matrix $V$ and an $n \times m$ diagonal matrix $\Sigma$ with non-negative real diagonal entries such that $A = U\Sigma V^*$.

Proof: It is enough to prove the result for the complex case since the real one is a special case of it. We can also assume without loss of generality that $n \geq m$ since $A = U\Sigma V^*$ implies $A^* = V\Sigma^T U^*$ (recall $\Sigma$ is real) so that we can switch easily between the $n \leq m$ case and the $n \geq m$ case by taking a conjugate transpose.

Let us first suppose that $\text{rank}(A) = m$. Then as we saw above, the matrix $A^*A$ is positive definite, so all of its eigenvalues are positive. Since $A^*A$ is self-adjoint, it is unitarily diagonalizable. Namely, there exists an orthonormal basis $\{v_1, v_2, \ldots, v_m\}$ of eigenvectors of $A^*A$ for $\mathbb{C}^m$. Say $A^*v_i = \lambda_i v_i$ where the eigenvalues may be repeated. We set $V$ to be the $m \times m$ matrix whose $i$th column is $v_i$. It is clear that $V$ is a unitary matrix. Let

$$
\sigma_i = \sqrt{\lambda_i}, \quad u_i = \frac{1}{\sigma_i} Av_i.
$$

We note that since each $\lambda_i$ is a positive real number, it has a unique positive square-root $\sigma_i$. Let $\Sigma$ be the $n \times m$ matrix such that $\Sigma_{ii} = \sigma_i$ and $\Sigma_{ij} = 0$ if $i \neq j$. Now, notice that $u_i$ is an $n \times 1$ column vector and

$$
\langle u_i, u_j \rangle = \left\langle \frac{1}{\sigma_i} Av_i, \frac{1}{\sigma_j} Av_j \right\rangle = \frac{1}{\sigma_i \sigma_j} \langle Av_i, Av_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle A^* Av_i, v_j \rangle = \frac{1}{\sigma_i \sigma_j} \langle \lambda_i v_i, v_j \rangle = \frac{\sigma_i^2}{\sigma_i \sigma_j} \langle v_i, v_j \rangle = \frac{\sigma_i^2}{\sigma_i \sigma_j} \delta_{ij}
$$

Then it is clear that $\{u_1, u_2, \ldots, u_m\}$ is an orthonormal set of vectors in $\mathbb{C}^n$. By the Gram-Schmidt procedure, complete this set to an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ of $\mathbb{C}^n$. If we set $U$ to be the $n \times n$ matrix whose $i$th column is $u_i$, then it is clear that $U$ is a unitary matrix.

Now, the relations $Av_i = \sigma_i u_i$ for $i = 1, \ldots, m$ and the fact that the last $n - m$ rows of $\Sigma$ are 0 give us that $AV = U\Sigma$. 
Finally, noting that $V^* = V^{-1}$, we get the singular value decomposition $A = U \Sigma V^*$.

The proof for $\text{rank}(A) = r < m$ is quite similar: Copy the same procedure above for positive eigenvalues of $A$ and get \{$u_1, \ldots, u_r$\}. For the 0 eigenvalues, do not define $u_i$ by the formula but rather complete \{$u_1, \ldots, u_r$\} to an orthonormal basis for $\mathbb{C}^n$ by the Gram-Schmidt procedure. The rest of the proof is the same. $\square$

**Example:** Consider the following two by two real matrix

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}.$$  

Let us find a singular value decomposition for $A$. We will essentially follow the steps of the proof above. Notice that

$$A^T A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$$

Next, compute the eigenvalues and eigenvectors of $A$. We see that

$$\Delta_A(x) = (x - 4)(x - 13) - 36 = (x - 16)(x - 1)$$

therefore the eigenvalues are $\lambda_1 = 16$ and $\lambda_2 = 1$. We can immediately compute the singular values to be $\sigma_1 = \sqrt{\lambda_1} = 4$ and $\sigma_2 = \sqrt{\lambda_2} = 1$. By computing normalized eigenvectors for $A^T A$ we get

$$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$ 

Next compute $u_1 = \frac{1}{\sigma_1} Av_1$ and $u_2 = \frac{1}{\sigma_2} Av_2$:

$$u_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad u_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$ 

If we let $U$ to be the matrix with columns $u_1, u_2$, $V$ to be the matrix with columns $v_1, v_2$ and $\Sigma$ to be the diagonal matrix with diagonal entries $\sigma_1, \sigma_2$ we obtain the singular value decomposition

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = U \Sigma V^T = \begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \end{bmatrix}.$$ 

Note that $A$ is not a diagonalizable matrix itself, however it still has the singular value decomposition above.

**Exercises:** These exercises require a little bit of internet searching

1. Look up the definition of the “pseudoinverse” or “Moore-Penrose inverse” of an $n \times m$ matrix. Prove, by using the singular value decomposition, that every real or complex matrix has a pseudoinverse.

2. Search for “Applications of Singular Value Decomposition”. Don’t stop reading until you are amazed.
MATH 262 Lecture, 21 May 2020

1 Polar Decomposition

1.1 Existence

As a motivation for what is coming up, let us start from a very well known fact about complex numbers. Every complex number can be written as $z = a + ib$ where $a, b \in \mathbb{R}$ and $i^2 = -1$. But complex numbers also have a polar representation $z = re^{i\theta}$ where $r \geq 0$ is a real number and $\theta$ is a real number. Since $e^{i\theta} = e^{i(\theta + 2\pi)}$, the choice of $\theta$ is not unique and is only well-defined up to $2\pi$. By Euler’s formula, we know that $re^{i\theta} = r \cos(\theta) + ir \sin(\theta)$ and this formula allows us to go back and forth between the standard and polar representations. The existence of the polar representation of a complex number is of great use, both conceptually and computationally. To give one example, we can easily express $n$th roots in terms of the polar representation: $r^{1/n}e^{i\theta/n}$ is one $n$th root of $z$ and using the fact that $\theta$ is well-defined only up to $2\pi$, we can find the others to be of the form $r^{1/n}e^{i(\theta + 2\pi k)/n}$ where $k$ is an integer.

The question that we want to address in this lecture is whether or not we can generalize the polar representation of a complex number to any $n \times n$ complex matrix. We first have to find the correct analogues of $r$ and $e^{i\theta}$ in the case of $n \times n$ matrices. The key property of $r$ is that it is both real and non-negative. If we regard the correct matrix analogue of $r$ to be a matrix whose eigenvalues are both real and non-negative, then a natural candidate is a non-negative definite matrix, since every such matrix is Hermitian and has non-negative eigenvalues. Likewise, the key property of $e^{i\theta}$ is that it lies on the unit circle. Therefore, if we regard the correct matrix analogue of $e^{i\theta}$ to be a matrix whose eigenvalues are all on the unit circle, then unitary matrices come to mind. So, with some stretch of imagination, we can formulate the question as follows: Given an $n \times n$ complex matrix $A$, can we find a non-negative definite $n \times n$ matrix $P$ and a unitary $n \times n$ matrix $U$ such that $A = UP$? The following theorem says that this optimistic guess works out:

**Theorem 1.** Let $A$ be an $n \times n$ complex matrix. Then there exists a unitary $n \times n$ matrix $U$ and a non-negative definite matrix $P$ such that $A = UP$. If $A$ is real, then both $U$ and $P$ can be taken to be real (so in this case $U$ is an orthogonal matrix and $P$ is a symmetric matrix with non-negative eigenvalues).

**Proof:** By the results of the previous lecture, $A$ has a singular value decomposition. Namely, there exist $n \times n$ unitary matrices $V, W$ and an $n \times n$ diagonal matrix $\Sigma$ with non-negative entries such that $A = V\Sigma W^*$. Setting $U = VW^*$ and $P = W\Sigma W^*$, we get $A = V\Sigma W^* = VW^*W\Sigma W^* = UP$.

We claim that $U$ is unitary and $P$ is non-negative definite. Indeed, it is easy to check that the product of two unitary matrices is unitary and since $U = VW^*$, we conclude that $U$ is unitary. On the other hand, $\Sigma$ has non-negative real eigenvalues. Since $P$ is unitarily similar to $\Sigma$, it has the same eigenvalues and it is also self-adjoint. Therefore $P$ is non-negative definite.
If $A$ is real, then by the singular value decomposition of real matrices, $V$ and $W$ above are real and therefore orthogonal. This implies that $U = VW^T$ is orthogonal. Since $\Sigma$ is always real anyway, $P = W\Sigma W^T$ is also real. This finishes the proof. □

**Definition 1.** The decomposition $A = UP$ of an $n \times n$ complex matrix as a product of a unitary $n \times n$ matrix and a non-negative definite $n \times n$ matrix is called a **polar decomposition** of $A$.

### 1.2 Uniqueness

Is the polar decomposition unique? Even in the $1 \times 1$ case, namely the case of complex numbers, we need to be a little bit careful: Say $z = re^{i\theta}$. Then $r$ is always uniquely determined by $z$. But the factor $e^{i\theta}$ (or $\theta$ up to $2\pi$) is uniquely determined iff $z \neq 0$. For the case $z = 0$, there is no uniqueness. The analogue of $z \neq 0$ for matrices would be invertibility. Our goal is to prove the uniqueness of polar decomposition for invertible matrices.

**Lemma 1.** (a) The inverse of a positive definite matrix is positive definite.

(b) Every positive definite matrix has a positive definite $n$th root for any positive integer $n$. Namely, if $P$ is a positive definite matrix, then there exists a positive definite matrix $Q$ such that $Q^n = P$.

(c) The product of two positive definite matrices has positive real eigenvalues (but it doesn’t have to be positive definite).

**Proof:**

(a) Suppose that $P$ is a positive definite matrix. Then it is unitarily diagonalizable with positive eigenvalues. Write $P = U\Sigma U^*$ where $U$ is unitary and $\Sigma$ is diagonal with positive diagonal entries. Then, clearly $P^{-1} = (U^*)^{-1}\Sigma^{-1}U^{-1} = U\Sigma^{-1}U^*$. The matrix $\Sigma^{-1}$ is diagonal with all diagonal entries positive. This implies that $P^{-1}$ is unitarily similar to a diagonal matrix with all diagonal entries positive real, so $P^{-1}$ is positive definite.

(b) As in part (a), say $P = U\Sigma U^*$ where $U$ is unitary and $\Sigma$ is diagonal with positive diagonal entries. Set $Q = U\Sigma^{1/n}U^*$ where in $\Sigma^{1/n}$ we simply take the positive real $n$th root of each diagonal entry. Then it is clear that $Q$ is positive definite. We have

$$Q^n = (U\Sigma^{1/n}U^*)^n = U(\Sigma^{1/n})^nU^* = P$$

establishing the claim.

(c) Say $P_1$ and $P_2$ are both positive definite. Let $Q$ be a positive definite square-root of $P_2$, which exists by part (b). Then

$$P_1P_2 = P_1Q^2 = Q^{-1}(QP_1Q)Q.$$

This relation implies first of all that $P_1P_2$ is similar to $QP_1Q$, therefore they have the same set of eigenvalues. On the other hand, $QP_1Q$ is positive definite: We have $(QP_1Q)^* = Q^*P_1^*Q^* = QP_1Q$ so it is self-adjoint. Also, for any vector $x$, we have

$$x^TQP_1Qx = (Qx)^T P_1(Qx) \geq 0$$

in view of the fact that $P_1$ is positive definite. Moreover, since $Q$ is invertible, equality can happen iff $x = 0$. This shows that $QP_1Q$ is positive definite. Hence, the eigenvalues of $QP_1Q$ are real and positive. Therefore the eigenvalues of $P_1P_2$ are real and positive. (An example where $P_1P_2$ is not self-adjoint, so that it fails to be positive definite, can be easily found.) □
Proposition 1. Let $A$ be an $n \times n$ invertible complex matrix. Then the polar decomposition of $A$ is unique. In other words, there exists a unique unitary $n \times n$ matrix $U$ and a unique non-negative definite matrix $P$ such that $A = UP$. Furthermore, $P$ is positive definite.

Proof: If $A$ is invertible and $A = UP$, then $\det(P) = \det(A)/\det(U)$ is non-zero, therefore $P$ has strictly positive eigenvalues. Hence the $P$ in any polar decomposition of an invertible matrix must be positive definite.

Suppose now that $A = U_1P_1 = U_2P_2$ are two polar decompositions of $A$ where $U_1, U_2$ are unitary and $P_1, P_2$ positive definite. Now, write
\[ U_2^{-1}U_1 = P_2P_1^{-1}. \]

The product on the left hand side is a unitary matrix, therefore it is unitarily diagonalizable with all its eigenvalues on the unit circle. For the right hand side, $P_1^{-1}$ is positive definite by part (a) of the lemma, and the product $P_2P_1^{-1}$ has positive real eigenvalues by part (c) of the lemma. The only number on the unit circle which is positive and real is 1. Therefore, all eigenvalues of $U_2^{-1}U_1$ are 1. The only diagonalizable matrix with all eigenvalues equal to 1 is the identity matrix. Hence $U_2^{-1}U_1 = I$ and $P_2P_1^{-1} = I$. This shows us that $U_1 = U_2$ and $P_1 = P_2$, hence the polar decomposition is unique. $\square$
These notes are written with an intention to clarify some of the points in the lectures related to multilinear algebra.

1 Free vector space over a set

Let $S$ be any set (finite or infinite). Assume that we don’t have any extra structure on $S$ (for instance, no binary operations), and if there are some, we simply forget them. Let $F$ be a field.

**Definition 1.** The free vector space $F(S)$ over $S$, as a set, is the set of all finite formal linear combinations of elements of $S$. More explicitly,

$$S = \{c_1 \cdot s_1 + c_2 \cdot s_2 + \ldots + c_m \cdot s_m|c_i \in F, s_i \in S\}$$

Here, $m$ is not fixed and can take any nonnegative integer value. The elements of $S$ that do not appear in such an expression are assumed to have coefficient 0.

Addition and scalar multiplication on $F(S)$ are defined as follows: Say $v = \sum c_i \cdot s_i$ and $w = \sum d_i \cdot s_i$ are two elements of $F(S)$, where $c_i, d_i \in F$. Then,

$$v + w = \sum (c_i + d_i) \cdot s_i.$$  

Similarly, for $c \in F$,

$$cv = \sum (cc_i) \cdot s_i.$$  

**Remark:** The previous definition can be made more rigorous as follows: Let $F(S)$ be the set of all functions $f : S \rightarrow F$ with finite support, in other words all functions which have non-zero values only at finitely many elements of $S$. Then, the operations are the usual addition of functions and scalar multiplication of a function with an element of $F$.

**Exercise:** Prove that $F(S)$ is a vector space. (Hint: Either check all the axioms directly, or show that the definition is equivalent to the one in the remark, then show that such functions form a subspace of all functions from $S$ to $F$.)

**Example:** Let $F = \mathbb{R}$ and $S = \{1, 3, 4\}$. Recall that we will treat these three elements as if they are not related at all; just like three arbitrary symbols. All elements of $F(S)$ are of the form

$$c_1 \cdot 1 + c_2 \cdot 3 + c_3 \cdot 4 \quad c_1, c_2, c_3 \in \mathbb{R}.$$  

For example, the following equality holds in $F(S)$:

$$(2 \cdot 1 + 0 \cdot 3 + 7 \cdot 4) + (5 \cdot 1 - 2 \cdot 3 + \sqrt{2} \cdot 4) = 7 \cdot 1 - 2 \cdot 3 + (7 + \sqrt{2}) \cdot 4.$$  

On the other hand,

$$1 \cdot 1 + 1 \cdot 3 \neq 1 \cdot 4.$$  

The zero vector can be written in many different ways, and these elements are equal: for instance

$$0 \cdot 1, \quad 0 \cdot 1 + 0 \cdot 4, \ldots$$  

since the coefficients which are not shown are assumed to be zero.

**Exercise:** Show that $S$ is a basis for $F(S)$. (This exercise is more or less definitionnal.)

The result of this exercise also immediately implies the following: The vector space $F(S)$ is finite dimensional if and only if $S$ is a finite set.
Proposition 1. Let $V$ be a vector space and let $f : S \rightarrow V$ be any function. Then, there is a unique linear transformation $T : F(S) \rightarrow V$ such that $T(1 \cdot s) = f(s)$ for all $s \in S$.

Sketch of proof: Let us define $T$ as follows

$$T(\sum c_i \cdot s_i) = \sum c_i \cdot f(s_i).$$

It is clear that $T(1 \cdot s) = f(s)$ for all $s \in S$. One then has to check routinely that $T$ satisfies the conditions for being a linear transformation.

Definition 2. The linear transformation $T$ above is said to be an extension of $f$ to $F(S)$. (Sometimes, with abuse of notation, $T$ itself is denoted again by $f$.)

2 Digression on Quotients and Quotient Vector Spaces

2.1 Quotients in General

First, let us recall the general idea of a quotient construction. Let $A$ be a set, and let $\sim$ be an equivalence relation on $A$. By this we mean,

- $\sim$ is reflexive: $a \sim a$ for all $a \in A$.
- $\sim$ is symmetric: If $a \sim b$, then $b \sim a$.
- $\sim$ is transitive: If $a \sim b$ and $b \sim c$, then $a \sim c$.

The equivalence class of an element of $A$ is defined to be:

$$[a] = \{ x \in A | x \sim a \}$$

Then it is a routine exercise to check that $A$ is a disjoint union of equivalence classes. The set of equivalence classes is denoted by $A/\sim$ and called the quotient of $A$ by this equivalence relation.

Suppose that $B$ is another set and $f : A \rightarrow B$ is a function. Then we say that $f$ induces a well-defined function $\overline{f}$ on $A/\sim$ if $a \sim b$ implies $f(a) = f(b)$. In that case, $\overline{f}$ is defined by the rule

$$\overline{f}(\overline{a}) = f(a).$$

The essential point is that $f$ must be constant within any equivalence class. Otherwise $\overline{f}$ is not well-defined, and doesn’t make sense. Similarly, a binary operation on $A$ induces a well-defined binary operation on $A/\sim$ if the equivalence class of the result doesn’t change when we replace each element undergoing the operation by an equivalent element.

Example: The prototypical example of a quotient construction is modular arithmetic. Suppose that $n$ is a positive integer. Define an equivalence relation on $\mathbb{Z}$ by saying

$$a \sim b \text{ if } a - b \text{ is divisible by } n$$

It is a straightforward exercise that $\sim$ is indeed an equivalence relation. The equivalence classes are usually referred to as “numbers modulo $n$”. Notice that $\overline{a} = \overline{a + kn}$ for any integer $k$. The set of equivalence classes $\mathbb{Z}/\sim$ (which is usually denoted by $\mathbb{Z}/n\mathbb{Z}$ instead) has exactly $n$ elements.
Continuing the example, the binary operations of addition and multiplication on \( \mathbb{Z} \) induce well-defined operations on \( \mathbb{Z}/n\mathbb{Z} \). To check this, one needs to verify that replacing a number, which is one of the operands, by another number in the same equivalence class does not change the equivalence class of the result (This is why “modular arithmetic” makes sense). As another example, the function \( \mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \) taking \( a \) to \( \overline{a} \) induces a well defined function \( \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z} \) (which is just the identity function). However, the function \( g : \mathbb{Z} \rightarrow \{-1, 0, +1\} \) defined such that \( g(a) = -1 \) if \( a < 0 \), \( g(a) = 0 \) if \( a = 0 \), \( g(a) = +1 \) if \( a > 0 \) does not induce a well-defined function \( \overline{g} : \mathbb{Z}/n\mathbb{Z} \rightarrow \{-1, 0, +1\} \).

2.2 Quotient Vector Spaces

Let \( V \) be a vector space over a field, and let \( W \) be any subspace. We want to form the “quotient vector space” \( V/W \), which will just be an example of the quotient construction that was just discussed. But, in addition, \( V/W \) itself will have vector space structure.

**Definition 3.** Let \( V \) be a vector space over a field \( F \), and \( W \) a subspace of \( V \). We define a relation \( \sim \) on \( V \) as follows:

\[
v \sim u \Leftrightarrow v - u \in W.
\]

**Proposition 2.** The relation \( \sim \) is an equivalence relation on \( V \).

**Sketch of proof:** Reflexivity holds since \( v - v = 0 \in W \) for any vector \( v \). Symmetry is a consequence of the fact that \( W \) is closed under multiplication by \(-1\), and transitivity is a consequence of the fact that \( W \) is closed under vector addition.

The quotient space \( V/\sim \) will be denoted by \( V/W \) from now on. Its elements will either be denoted by \( \overline{v} \), or by \( v + W \), interchangeably. The second notation makes sense since indeed

\[
\overline{v} = \{v + w \mid w \in W\}.
\]

Notice that \( \overline{0} = 0 \) in \( V/W \) if and only if \( v \in W \).

**Proposition 3.** The vector addition and scalar multiplication on \( V \) induce well-defined operations on \( V/W \), and with these operations \( V/W \) is a vector space over \( F \).

**Sketch of proof:** One needs to check the following statements for well-definedness: If \( \overline{v_1} = \overline{v_2} \) and \( \overline{w_1} = \overline{w_2} \) then \( \overline{v_1 + v_1} = \overline{v_2 + w_2} \). If \( c \in F \) and \( \overline{v_1} = \overline{v_2} \), then \( \overline{cv_1} = \overline{cv_2} \). These statements are consequences of the fact that \( W \) is closed under addition and scalar multiplication. Checking that \( V/W \) is a vector space can be done by a routine verification of the axioms.

**Proposition 4.** Suppose that \( T : V \rightarrow U \) is a linear transformation and \( W \) is a subspace of \( V \). Then \( T \) induces a well-defined linear transformation \( \overline{T} : V/W \rightarrow U \) if and only if \( T(w) = 0 \) for all \( w \in W \).

**Proof:** By the general discussion in the quotient construction above, \( T \) induces a well-defined function on \( V/W \) iff it is constant on equivalence classes, namely \( v_1 \sim v_2 \) implies \( T(v_1) = T(v_2) \). On the other hand, \( v_1 \sim v_2 \) if and only if \( v_1 = v_2 + w \) for some \( w \in W \). Since \( T \) is a linear transformation, in this case the equation \( T(v_1) = T(v_2) \) will hold if and only if \( T(w) = 0 \). The linearity of the induced function \( \overline{T} \) directly follows from the formula \( \overline{T}(\overline{v}) = \overline{T(v)} \).

**Remark:** Notice that throughout this discussion, \( V \) and \( W \) were not assumed to be finite dimensional. Indeed, the tensor product construction below will be most interesting in the infinite dimensional case. Therefore, this generality is crucial for this discussion.
3 Tensor Products of Vector Spaces

The goal is: Given two vector spaces $V$ and $W$ over a field $F$, to construct a new vector space $V \otimes W$ such that the vector space operations are like those of $V$ and $W$ on each component, but we don’t do the operations simultaneously. Instead, it behaves like one of these operations on a given component when the other component is fixed. (e.g. As an analogy, think of algebraic operations such as $x \cdot y + (2x) \cdot y$, which is equal to $(3x) \cdot y$ but not equal to $(3x) \cdot (2y)$. We want to cook up a vector space version of this.)

The strategy for constructing $V \otimes W$ is quite roundabout, which usually causes some confusion. First, look at the direct product $V \times W$. If we use the immediate vector space structure on $V \times W$, it is useless for our purpose, since the operations will be simultaneous. Instead, rip up all the vector space structure on $V \times W$ and treat it just as a set. Then look at the free vector space $\mathcal{F}(V \times W)$. This vector space infinite dimensional (unless we are working in the special case of finite fields and finite dimensional vector spaces), and essentially with no interesting relations among its elements. To get what we want, we need to impose back the relations that we would want to hold manually. This can be done by the quotient construction, and ultimately $V \otimes W$ will be defined as a certain quotient vector space of $\mathcal{F}(V \times W)$. Now let us write this down mathematically.

**Definition 4.** Let $V$ and $W$ be vector spaces over a field $F$. Then $V \otimes W$ is the quotient vector space

$$V \otimes W = \mathcal{F}(V \times W)/U$$

where $U$ is the subspace of $\mathcal{F}(V \times W)$ which is spanned by the following elements:

1. $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$, \hspace{1em} $v_1, v_2 \in V, w \in W$
2. $(v, w_1 + w_2) - (v, w_1) - (v, w_2)$, \hspace{1em} $v \in V, w_1, w_2 \in W$
3. $(cv, w) - c \cdot (v, w)$, \hspace{1em} $c \in F, v \in V, w \in W$
4. $(v, cw) - c \cdot (v, w)$, \hspace{1em} $c \in F, v \in V, w \in W$

Let us elaborate a little bit on what the elements of $V \otimes W$ look like. First of all, elements of $\mathcal{F}(V \times W)$ are formal finite sums

$$\sum c_i \cdot (v^{(i)}, w^{(i)})$$

where $c_i \in F$, $v^{(i)} \in V$ and $w^{(i)} \in W$. Denote $\overline{(v, w)}$ by $v \otimes w$ and more generally $\sum c_i \cdot (v^{(i)}, w^{(i)})$ by $\sum c_i \cdot v^{(i)} \otimes w^{(i)}$. Hence, any element of $V \otimes W$ can be written in this form.

**Exercise:** Take $F = \mathbb{R}$, $V = W = \mathbb{R}^2$ with their usual vector space structure. Which elements of $V \otimes W$ can be written in the form $v \otimes w$ with $v \in V, w \in W$ (without a sum)?

The elements of $U$ are in the equivalence class of 0 in $\mathcal{F}(V \times W)$. Therefore, the elements of the spanning set of $U$ induce the following relations in $V \otimes W$:

1. $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$, \hspace{1em} $v_1, v_2 \in V, w \in W$
2. $v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$, \hspace{1em} $v \in V, w_1, w_2 \in W$
3. $(cv) \otimes w = c(v \otimes w)$, \hspace{1em} $c \in F, v \in V, w \in W$
4. \( v \otimes (cw) = c(v \otimes w), \quad c \in F, v \in V, w \in W \)

Remark: The relations above could be used to give some kind of “layman’s definition” to tensor product: “It is a vector space subject to the rules above, and their consequences”. Of course, this wouldn’t be satisfactory from a rigorous point of view, since we wouldn’t then be sure whether such a space exists, and how to get a grip on its properties if it exists.

Proposition 5. Suppose that \( V \) and \( W \) are finite dimensional vector spaces of dimensions \( m \) and \( n \) respectively. Suppose that \( \{v_1, v_2, \ldots, v_m\} \) is a basis for \( V \) and \( \{w_1, w_2, \ldots, w_n\} \) is a basis for \( W \). Then \( B = \{v_i \otimes w_j\} \) where \( i \in \{1, 2, \ldots, m\} \) and \( j \in \{1, 2, \ldots, n\} \) is a basis for \( V \otimes W \). In particular, \( \dim(V \otimes W) = mn \).

Proof: Let us first show that \( B \) spans \( V \otimes W \). Since any element of \( V \otimes W \) is of the form \( \sum c_i \cdot v^{(i)} \otimes w^{(i)} \), it will be enough to show that for any \( v \in V, w \in W \) we have \( v \otimes w \in \text{Span}(B) \).

But we can write \( v = a_1v_1 + \ldots + a_mv_m \) and \( w = b_1w_1 + \ldots + b_nw_n \) for some \( a_i, b_j \in F \). Therefore, by repeatedly applying the relations in the tensor product, we get

\[
  v \otimes w = \sum a_i b_j (v_i \otimes w_j).
\]

Therefore \( v \otimes w \in \text{Span}(B) \).

Next, we need to show that \( B \) is linearly independent. As a preliminary step, we want to construct certain linear transformations \( \varphi_1, \ldots, \varphi_m \) from \( V \otimes W \) to \( W \). First, we define \( \varphi_i : V \times W \rightarrow W \) as the following ordinary function: Suppose that \( v = a_1v_1 + \ldots + a_mv_m \) and \( w = b_1w_1 + \ldots + b_nw_n \) for some \( a_i, b_j \in F \). Then let

\[
  \varphi_i(v, w) = a_i w
\]

Now, there is a unique extension of \( \varphi_i \) to a linear transformation \( F(V \times W) \rightarrow W \), which we will denote by the same letter \( \varphi_i \). We claim that \( \varphi_i \) induces a well-defined linear transformation \( V \otimes W \rightarrow W \), which will again be denoted by the same letter. In order to show this, we must check that \( \varphi_i \) is 0 on any element of \( U \) that appears in the quotient defining \( V \otimes W \). This can be checked one by one for all the four types of spanning elements.

Having decided that \( \varphi_i \) are well-defined linear transformations, let us finish the proof of linear independence of \( B \). Suppose that \( \sum_{i,j} a_{ij} (v_i \otimes w_j) = 0 \). Then applying \( \varphi_i \) to this expression, we find that for each value of \( i \),

\[
  \sum_j a_{ij} w_j = 0.
\]

But then by the independence of \( w_j \)'s, all \( a_{ij} \) must be zero. Since this is true for any value of \( i \), our proof is complete.
MATH 262
Practice Problems for Midterm I

The midterm 1 topics for MATH 262 will be probably all the content in the syllabus upto the end of “minimal polynomial and characteristic polynomial” (and not including “inner product spaces”). I will make an announcement in class when it becomes certain.

Practice Problems:
1. Let $V = \mathbb{R}^3$. Suppose that $\cdot$ represents the dot product and $\times$ represents the cross product of vectors in $\mathbb{R}^3$. Show that $f$ defined by
   \[ f(v_1, v_2, v_3) = v_1 \cdot (v_2 \times v_3) \]
   is an element of the dual space of $\bigwedge^3 \mathbb{R}^3$. Use this to show that $f(v_1, v_2, v_3)$ is equal to the determinant of the matrix with rows $v_1, v_2, v_3$.

2. For each of the items below, decide whether the statement is true or false. Explain your reasoning:
   (a) The determinant of a real, symmetric matrix is always nonnegative.
   (b) If $A$ is a $2 \times 3$ real matrix, then $\det(AA^T) = 0$.
   (c) If $A$ is a $3 \times 2$ real matrix, then $\det(AA^T) = 0$.
   (d) If $A$ is real, square matrix, then $\det(AA^T) \geq 0$.

3. Suppose that a square matrix $A$ can be written in the form
   \[ A = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix} \]
   where $B$ and $C$ are square matrices and $0$ denotes the zero matrix. Prove that
   \[ \det(A) = \det(B) \det(C). \]

4. Suppose that $A$ is a $2n \times 2n$ matrix which can be written in the form
   \[ A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \]
   where $B, C, D, E$ are all $n \times n$ matrices. Give an example to show that
   \[ \det(A) \neq \det(B) \det(E) - \det(C) \det(D) \]
   in general.

5. Let $\sigma$ be the permutation in $\text{Sym}(n)$ such that $\sigma(i) = n + 1 - i$ for each $i \in \{1, 2, \ldots, n\}$. Find $\text{sgn}(\sigma)$.

6. Let $A$ be a $6 \times 6$ tridiagonal matrix, namely, $a_{ij} = 0$ if $|i - j| > 1$. List all the permutations in $\text{Sym}(6)$ which possibly give a nonzero term in the determinant of $A$ and find their signs. Express the determinant in terms of the entries $a_{ij}$ with $|i - j| \leq 1$. 
7. Find the quotient and the remainder when \( f(x) = x^4 - x + 1 \) is divided by \( g(x) = x - 1 \) in \( \mathbb{F}_3[x] \).

8. Find a greatest common divisor \( d(x) \) of the three polynomials \( f_1(x) = 4x^2 + 2x, f_2(x) = 2x^3 + x^2 - 2x - 1, f_3(x) = 2x^4 + x^3 - 2x^2 + x + 1 \) in \( \mathbb{Q}[x] \). Find polynomials \( g_1, g_2, g_3 \in \mathbb{Q}[x] \) such that \( g_1f_1 + g_2f_2 + g_3f_3 = d \).

9. Write \( x^4 + 1 \) as a product of irreducible polynomials over \( \mathbb{C} \), over \( \mathbb{R} \) and over \( \mathbb{Q} \).

10. Let \( F \) be a field and \( J_1, J_2 \) two ideals of \( F[x] \).
   (a) Show that \( J_1 \cap J_2 \) is an ideal of \( F[x] \). If \( J_1 = (f_1) \) and \( J_2 = (f_2) \) what can you say about generators of \( J_1 \cap J_2 \)?
   (b) Let \( J_1 + J_2 = \{ f + g | f \in J_1, g \in J_2 \} \). Show that \( J_1 + J_2 \) is an ideal of \( F[x] \). If \( J_1 = (f_1) \) and \( J_2 = (f_2) \) what can you say about generators of \( J_1 + J_2 \)?
   (c) Show that \( J_1 \cup J_2 \) is not necessarily an ideal of \( F[x] \).

11. Find all eigenvalues and eigenspaces of the following matrices:
   \[
   (a) \begin{bmatrix}
   0 & 1 & 0 & 0 \\
   0 & 0 & 1 & 0 \\
   0 & 0 & 0 & 1 \\
   0 & 0 & 0 & 0
   \end{bmatrix}
   \]
   \[
   (b) \begin{bmatrix}
   3 & 0 & -1 \\
   -1 & 1 & 2 \\
   0 & 0 & 4
   \end{bmatrix}
   \]

12. Find all eigenvalues and eigenspaces for the operator \( T : \mathbb{R}^5 \to \mathbb{R}^5 \) defined by
   \[
   T(x_1, x_2, x_3, x_4, x_5) = (x_1 + 4x_2, -x_1 + x_2, x_2 - x_3, -3x_4 + 3x_5, x_4 - x_5).
   \]
   Also solve the problem if \( T \) is viewed as an operator on \( \mathbb{C}^5 \) instead.

13. Suppose that \( A \) is a square matrix over a field, such that the sum of all entries on each row is 0. Find an eigenvector of \( A \). Find \( det(A) \).

14. Let \( P : V \to V \) be a projection operator (recall that this means \( P^2 = P \)). Show that \( Ker(P), Im(P), Ker(P - I), Im(P - I) \) are all \( P \)-invariant subspaces of \( V \).

15. Let \( F \) be a field and \( N : F^n \to F^n \) be a nilpotent operator of index \( n \) (recall that this means: \( N^n = 0 \) but \( N^{n-1} \neq 0 \)). Describe all invariant subspaces of \( N \).

16. Find all eigenvalues and eigenspaces of the following matrices. Determine whether they are diagonalizable or not. If so, find an invertible matrix \( P \) and a diagonal matrix \( D \) such that the matrix equals \( PDP^{-1} \).
   \[
   (a) \begin{bmatrix}
   1 & -4 & 0 & 0 & 0 \\
   -1 & 1 & 0 & 0 & 0 \\
   -1 & 2 & -1 & 0 & 0 \\
   0 & 0 & 0 & -3 & 3 \\
   0 & 0 & 0 & 1 & -1
   \end{bmatrix}
   \]
17. Compute \[
\begin{bmatrix}
4 & -1 \\
-1 & 4
\end{bmatrix}^{262}.
\]

18. Find the characteristic polynomial and the minimal polynomial of the matrix
\[
\begin{bmatrix}
2 & 0 & 0 & 0 \\
5 & 2 & 0 & 0 \\
-7 & 1 & 1 & -1 \\
8 & -2 & 2 & 4
\end{bmatrix}.
\]

19. Suppose that the characteristic polynomial of an \(n \times n\) matrix \(A\) is \(\Delta_A(x) = x^n - x + 1\). Show that \(A\) is invertible and \(A^{-1} = I - A^{n-1}\).

20. Let \(V = M_{n \times n}(F)\) and \(T : V \rightarrow V\) is defined by \(T(A) = A^T\).
   (a) Show that \(T\) is a linear operator on \(V\).
   (b) Find the eigenvalues and eigenvectors of \(T\).
   (c) Show that if \(\text{char}(F) \neq 2\) then \(T\) is diagonalizable.

21. If \(A\) is a diagonalizable matrix over the field \(F\), show that all matrices in the set \(\{f(A) | f \in F[x]\}\) are simultaneously diagonalizable.
MATH 262 Practice Problems for Quiz 3 and 4

Below are some practice problems for Quiz 3 and Quiz 4. The topics covered are: Simultaneous diagonalization, inner product spaces, orthogonal and orthonormal bases, Gram-Schmidt orthogonalization process, the method of least squares, Fourier series, self adjoint operators, unitary and normal operators, spectral theorem. (Quadratic forms which were initially announced for quiz 4 will be postponed to quiz 5).

1. Show that the following real matrices are simultaneously diagonalizable and find an invertible matrix $P$ diagonalizing them:

\[
A = \begin{bmatrix}
-2 & -1 & -5 \\
0 & 0 & 0 \\
30 & 15 & -3
\end{bmatrix}, \quad
B = \begin{bmatrix}
-4 & -2 & 2 \\
0 & 0 & 0 \\
-12 & -6 & 6
\end{bmatrix}, \quad
C = \begin{bmatrix}
-11 & -4 & 4 \\
12 & 5 & -4 \\
-12 & -4 & 5
\end{bmatrix}
\]

2. Let $\mathcal{F}$ be a family of simultaneously diagonalizable $n \times n$ matrices over a field. Show that if $\mathcal{F}$ contains a matrix $A$ with $n$ distinct eigenvalues, then every eigenvector of $A$ is an eigenvector for all members of $\mathcal{F}$. Give a counterexample to show that this statement is not necessarily true if $A$ does not have $n$ distinct eigenvectors.

3. Show that the following formula defines an inner product on $\mathbb{R}^2$:

\[
\langle (x_1, y_1), (x_2, y_2) \rangle = 3x_1y_1 - 2x_2y_1 - 2x_1y_2 + 2x_2y_2.
\]

4. Show that if $\langle , \rangle_1$ and $\langle , \rangle_2$ are two inner products on a vector space $V$, then their sum $\langle , \rangle_1 + \langle , \rangle_2$ is another inner product. 1

5. Prove the parallelogram law for inner product spaces:

\[
||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2
\]

for all $x, y \in V$.

6. Consider $\mathbb{C}^3$ with the standard Hermitian inner product on it. Let $W = Span((1, i, 0), (1 - i, 2, 4i))$. Find an orthonormal basis for $W$. Find the orthogonal projection of $v = (6 + i, 4i, -4)$ along $W$.

7. Let $V$ be the vector space of real valued continuous functions on $[0, \pi]$, viewed as a real vector space, with the integral inner product

\[
\langle f, g \rangle = \int_0^\pi f(t)g(t)dt.
\]

Let $W = Span(sin(t), cos(t), 1, t)$. Find an orthonormal basis for $W$.

8. Use the method of least squares in order to find a linear function which best fits to the points \{(1, 2), (3, 4), (5, 7), (7, 9), (9, 12)\}.

9. Assume $f \in C[a, b]$, equipped with the integral inner product. The average value of $f$ over $[a, b]$ is defined to be

\[
\frac{1}{b-a} \int_a^b f(t)dt.
\]

Show that the average value of $f$ is equal to its projection along the function 1.
10. Let $V$ be a finite dimensional inner product space or a Hermitian inner product space. Suppose that $W$ is a subspace of $V$ and $W^\perp$ its orthogonal complement. Prove that
\[ \dim(W) + \dim(W^\perp) = \dim(V). \]

11. Show that if $TT^* = 0$ then $T = T^* = 0$.

12. Let $V$ be a finite dimensional Hermitian inner product space and $T$ a linear operator on $V$. Prove that $T$ is normal if and only if $||Tv|| = ||T^*v||$ for every $v \in V$.

13. Let $V = M_{n \times n}(\mathbb{C})$ equipped with $\langle A, B \rangle = Tr(AB^*)$.
   (a) Show that $\langle , \rangle$ is a Hermitian inner product on $V$.
   (b) Suppose that $P$ is a unitary matrix. Show that the operator $T$ on $V$ defined by $T(A) = PAP^*$ is a unitary linear transformation.

14. Suppose that $T$ is a self-adjoint operator on an $n$ dimensional inner product space, with $k$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. What is the minimal polynomial of $T$?

15. Let
\[ A = \begin{bmatrix} 1 + i & -1 + i \\ -1 + i & 1 + i \end{bmatrix}. \]
   Verify that $A$ is normal and find a unitary matrix $U$ such that $U^*AU$ is diagonal.

16. Find an orthonormal basis for $\mathbb{R}^3$ such that the matrix of the operator
\[ T(x, y, z) = (-2x + y + z, x - 2y + z, x + y - 2z) \]
   is diagonal.

17. Let $V$ be a Hermitian inner product space and $T$ a linear operator on $V$ such that $\langle Tv, v \rangle \in \mathbb{R}$ for every $v \in V$. Show that $T$ is self-adjoint.

18. Let $V$ be the vector space of complex valued continuous functions on $[0, 1]$ equipped with the integral Hermitian inner product
\[ \langle f, g \rangle = \int_0^1 f(x)\overline{g(x)}dx. \]
   Let $h \in V$ and $T$ be the linear operator on $V$ defined by the formula $T(f) = hf$. Show that $T$ is a unitary operator if and only if $|h(x)| = 1$ for all $x$. 
1 Classifying Linear Operators up to Change of Basis

Let us now leave aside inner product spaces and return to an older story. Say $V$ is a finite dimensional vector space over a field $F$. Suppose that $T : V \to V$ is a linear operator. We know from previous lectures that the following conditions are equivalent:

- $T$ is diagonalizable.
- $V$ is a direct sum of eigenspaces of $T$.
- There exists a basis of eigenvectors of $T$ for $V$.
- The minimal polynomial of $T$ is a product of distinct linear factors.

In terms of matrices, we can make the following formulation: For a matrix $A \in M_{n \times n}(F)$ if one, therefore all of these conditions hold, then there exists an invertible matrix $P$ such that $A = PDP^{-1}$ and $D$ is a diagonal matrix.

Let us look at this result in somewhat more depth: The entries of $D$ are far from being arbitrary numbers; they are precisely the eigenvalues of $T$. If two diagonalizable operators $T_1$ and $T_2$ have precisely the same eigenvalues with the same multiplicities, say that $D$ is the diagonal matrix having these eigenvalues as diagonal entries in some order. Then there exists a basis $B_1$ of $V$ such that the matrix of $T_1$ with respect to $B_1$ is $D$, and another basis $B_2$ of $V$ such that the matrix of $T_2$ with respect to $B_2$ is $D$ as well. This implies that we can say $T_1$ and $T_2$ are equivalent up to a change of basis (as an exercise, make this precise by carefully defining the equivalence relation).

Also, let us phrase this in terms of matrices. Suppose that $A$ and $B$ are both diagonalizable matrices with the same eigenvalues, occurring with the same multiplicities. Then as above, there exists a common diagonal matrix $D$, and invertible matrices $P$ and $Q$ such that $A = PDP^{-1}$ and $B = QDQ^{-1}$. We see that $A = (P^{-1}Q)D(P^{-1}Q)^{-1}$ so that $A$ and $B$ are similar. Conversely, if $A$ and $B$ are similar we know that they have the same characteristic polynomial, so they have the same eigenvalues occurring with the same multiplicities. We make the following conclusion: We can classify all diagonalizable matrices up to similarity by looking at the list of their eigenvalues (together with their multiplicities).

These observations lead to the following natural questions: What happens if we remove the condition of diagonalizability in the previous statements? Namely, how can we classify all operators on a vector space, up to equivalence under a change of basis? Or how can we classify all $n \times n$ matrices over $F$ up to similarity? Give me some clear cut recipe that decides whether two given square matrices are similar or not. If we a priori know that both matrices are diagonalizable, we know how to do that: Compute eigenvalues and their multiplicities for both matrices, check if they agree with each other or not, and we are done. How about in general?

Before we start solving this problem, we would like to simplify one aspect of it. One reason for an operator not to be diagonalizable over $F$ may be that its eigenvalues do not belong to the field $F$. A typical example is a rotation of $\mathbb{R}^2$ about the origin with an angle different from a multiple of $\pi$. In this case the eigenvalues are not real, they are complex. So such a rotation is not diagonalizable over $\mathbb{R}$. From our perspective, this is somewhat an artificial problem which can be completely solved by enlarging the field. If we pass to the complexification and visualize
the vector space over \( \mathbb{C} \), the rotation is diagonalizable (this is just a special case of the spectral theorem applied to an orthogonal operator). It turns out that for an arbitrary field \( F \), if a polynomial \( f \in F[x] \) does not have any roots in \( F \), then one can always pass to a larger field \( K \), a certain extension of \( F \), in which \( f \) has a root. (We will not prove this statement since it requires some field theory.) So in some sense, the absence of an eigenvalue of \( T \) in \( F \) is an artificial obstacle that can be avoided by enlarging the field if necessary. For this lecture and the remaining ones we will assume that the characteristic polynomial of the operator \( T \) is a product of linear factors in \( F[x] \), so that we don’t have this problem.

2 Jordan-Chevalley Decomposition

Let \( V \) be a finite dimensional vector space over a field \( F \). Another name for a diagonalizable operator \( T \) on \( V \) is a semisimple operator. Also recall another definition we had for some time: \( T \) is called a nilpotent operator if \( T^m = 0 \) for some positive integer \( m \). The goal of this section is to prove that every linear operator whose characteristic polynomial is a product of linear factors in \( F[x] \) can be decomposed as the sum of a semisimple operator and a nilpotent operator, and under some additional natural requirements this decomposition is unique.

We first want to prove a result reminiscent of the fact that when \( T \) is diagonalizable \( V \) is equal to the direct sum of eigenspaces of \( T \):

**Lemma 1.** Let \( V \) be a finite dimensional vector space over \( F \) and \( T \) an operator on \( V \) with characteristic polynomial

\[
\Delta_T(x) = (x - \lambda_1)^{\mu_1}(x - \lambda_2)^{\mu_2}\cdots(x - \lambda_k)^{\mu_k}.
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_k \) are distinct elements of \( F \). Let \( C_i = \ker((T - \lambda_i I)^{\mu_i}) \) for \( i = 1, \ldots, k \). Then

\[
V = C_1 \oplus C_2 \oplus \cdots \oplus C_k.
\]

**Proof:** Let us start by some preparation. Set \( T_i = (T - \lambda_i I)^{\mu_i} \). Notice that the operators \( T_i \) commute, namely \( T_i T_j = T_j T_i \) for all \( i, j \) since all of them are polynomials in \( T \). Also, by the Cayley-Hamilton theorem \( \Delta_T(T) = 0 \), therefore

\[
T_1 T_2 \ldots T_k = 0.
\]

Define the polynomials

\[
p_i(x) = \frac{\Delta_T(x)}{(x - \lambda_i)^{\mu_i}}.
\]

Since the set of polynomials \( p_1(x), p_2(x), \ldots, p_k(x) \) have no common factor, the ideal generated by all of them must be the whole ring \( F[x] \). In particular, there exist polynomials \( q_1(x), q_2(x), \ldots, q_k(x) \in F[x] \) such that

\[
1 = q_1 p_1 + q_2 p_2 + \ldots + q_k p_k
\]

First of all, let us show that \( V = C_1 + C_2 + \ldots + C_k \). For this purpose, say \( v \in V \). Let \( v_i \) be defined by

\[
v_i = q_i(T)p_i(T)v
\]
Then, since \( T_i v_i = q_i(T)T_1p_i(T)v = q_i(T)T_1T_2 \ldots T_kv = 0 \), we see that \( v_i \in \ker(T_i) = C_i \). Also,

\[
v = q_1(T)p_1(T)v + \ldots + q_k(T)p_k(T)v = v_1 + \ldots + v_k
\]

and this shows that \( V = C_1 + C_2 + \ldots + C_k \).

To finish the proof, we must show that the sum is actually a direct sum. And to show this, it is enough to prove that for every \( i = 1, \ldots, k-1 \),

\[
(C_1 + C_2 + \ldots + C_i) \cap C_{i+1} = \{0\}
\]

So suppose that \( v \in (C_1 + C_2 + \ldots + C_i) \cap C_{i+1} \). Since \( v \in C_{i+1} \) we have \( T_{i+1}v = 0 \). Since \( v \in C_1+C_2+\ldots+C_i \), we also must have \( T_1T_2 \ldots T_i v = 0 \). But since the polynomials \( (x-\lambda_{i+1})^{\mu_{i+1}} \) and \( (x-\lambda_{i})^{\mu_{i}} \) are coprime, there must exist polynomials \( g(x), h(x) \in F[x] \) such that

\[
1 = g(x)(x-\lambda_{i+1})^{\mu_{i+1}} + h(x)(x-\lambda_{i})^{\mu_{i}} \ldots (x-\lambda_{i})^{\mu_{i}}.
\]

But then

\[
v = g(T)(T - \lambda_{i+1}I)^{\mu_{i+1}}v + h(T)(T - \lambda_{i}I)^{\mu_{i}} \ldots (T - \lambda_{i}I)^{\mu_{i}}v = 0
\]

which implies that \( v = 0 \). This finishes the proof. \( \square \)

**Lemma 2.** With the same notation as in the previous lemma, for each \( i \), \( C_i \) is an invariant subspace of \( T \).

**Proof:** Say \( v \in C_i = \ker((T - \lambda_i I)^{\mu_i}) \). We need to show that \( Tv \in C_i \). But clearly, \( T \) commutes with \( (T - \lambda_i I)^{\mu_i} \). So,

\[
(T - \lambda_i I)^{\mu_i} Tv = T(T - \lambda_i I)^{\mu_i}v = 0
\]

therefore \( Tv \in C_i \). \( \square \)

**Theorem 1.** (Jordan-Chevalley decomposition) Suppose that \( V \) is a finite dimensional vector space over a field \( F \). Let \( T \) be a linear operator on \( V \) and assume that the characteristic polynomial of \( T \) is a product of linear factors in \( F[x] \). Then there exists unique semisimple and nilpotent operators \( S, N \) commuting with each other and with \( T \), such that \( T = S + N \).

**Proof:** As in the lemma, let us assume that

\[
\Delta_T(x) = (x-\lambda_1)^{\mu_1}(x-\lambda_2)^{\mu_2} \ldots (x-\lambda_k)^{\mu_k}
\]

where \( \lambda_1, \ldots, \lambda_k \) are distinct eigenvalues of \( T \) lying in \( F \). Let \( C_i = \ker((T - \lambda_i I)^{\mu_i}) \) for \( i = 1, \ldots, k \). By the lemma, we now know that

\[
V = C_1 \oplus C_2 \oplus \ldots \oplus C_k.
\]

First, let us show the existence of operators \( S, N \) with the given properties. Let \( S \) be the unique linear operator on \( V \) such that for every \( v_i \in C_i \),

\[
S(v_i) = \lambda_i v_i.
\]

It is clear that \( S \) is semisimple: On each \( C_i \) it is equal to \( \lambda_i \) times identity, hence it is a diagonalizable operator in view of the fact that \( V = C_1 \oplus C_2 \oplus \ldots \oplus C_k \). Let \( N = T - S \). We
want to show that $N$ is nilpotent. First, suppose that $v_i \in C_i$. Then since $S = \lambda_i I$ on $C_i$, we have
\[ N^{\mu_i} v_i = (T - S)^{\mu_i} v_i = (T - \lambda_i I)^{\mu_i} v_i = 0. \]
Now, let $r = \max(\mu_1, \ldots, \mu_k)$. Then for any $v \in V$ we can write $v = v_1 + \ldots + v_k$ with $v_i \in C_i$ and
\[ N^r v = N^r (v_1 + v_2 + \ldots + v_k) = 0. \]
This shows us that $N$ is nilpotent.

We want to show that $T$, $N$ and $S$ all commute. Let $v_i \in C_i$. Noting again that on $C_i$, the operator $S$ agrees with $\lambda_i I$, we have
\[ TSv_i = T(\lambda_i v_i) = \lambda_i Tv_i = STv_i. \]
In the last equality, we used the fact that $C_i$ is an invariant subspace of $T$, so that $Tv_i \in C_i$ and $S$ acts by multiplication by $\lambda_i$ on it. Now, for a general $v \in V$, we can write $v = v_1 + v_2 + \ldots + v_k$ and
\[ TSv = TSv_1 + \ldots + TSv_k = STv_1 + \ldots + STv_k = STv. \]
Therefore, $S$ and $T$ commute. Now, $N = T - S$ and since it clearly follows that $T$ and $T - S$ commute, we deduce that $T$ and $N$ commute. Finally, $S = T - N$ so $N$ and $S$ commute. We finished the proof of existence.

Uniqueness: Suppose now that $T$, $N$ and $S$ are the operators in the first part of the proof, $S'$ is semisimple, $N'$ is nilpotent, and $T, S', N'$ all commute pairwise. Since $S'$ is semisimple we have a decomposition
\[ V = W_{\nu_1} \oplus W_{\nu_2} \oplus \ldots \oplus W_{\nu_l} \]
where $W_{\nu_1}, \ldots, W_{\nu_l}$ are the eigenspaces for $S'$ for the eigenvalues $\nu_1, \ldots, \nu_l$ respectively. Clearly, $S'$ restricted to $W_{\nu_i}$ agrees with $\nu_i I$. Since $T$ commutes with $S'$, we have for any $w_i \in W_{\nu_i}$,
\[ S'Tw_i = TS'w_i = T\nu_iw_i = \nu_i T w_i \]
so that $Tw_i$ is an eigenvector for $S'$ and therefore belongs to $W_{\nu_i}$. This implies that each $W_{\nu_i}$ is an invariant subspace of $T$. In a similar way, each $W_{\nu_i}$ is an invariant subspace of $N'$. Now, $N'$ is still nilpotent when restricted to $W_{\nu_i}$ and on this subspace we have $N' = T - S' = T - \nu_i I$. Therefore there exists an integer $c_i$ such that for every $w_i \in W_{\nu_i}$ we have $(T - \nu_i I)^{c_i} w_i = 0$. Now, since any nilpotent operator has a nontrivial kernel, there exists a non-zero vector $z_i \in W_{\nu_i}$ such that $N'z_i = 0$. But then for this vector we clearly have
\[ Tz_i = \nu_i z_i \]
which implies that such a $z_i$ must be an eigenvector of $T$. In particular $\nu_i$ must be equal to $\lambda_j$ for some $j$. Without loss of generality, say $\nu_i = \lambda_i$ (which incidentally shows that $l \leq k$ at this point). Now, rewriting the relation above, for every $w_i \in W_{\nu_i}$ we have
\[ (T - \lambda_i I)^{c_i} w_i = 0 \]
therefore, we conclude that $W_{\nu_i} \subseteq C_i$. Finally,
\[ \dim(V) = \dim(W_{\nu_1}) + \ldots + \dim(W_{\nu_l}) \leq \dim(C_1) + \ldots + \dim(C_l) \leq \dim(C_1) + \ldots + \dim(C_k) = \dim(V). \]
Therefore we must have equality at each step. This shows that $k = l$ and $W_{i_i} = C_i$ for every $i$. But then we get $S = S'$ since their restrictions to $C_i$ are both equal to $\lambda_i I$ for every $i$ and $V$ is a direct sum of $C_i$’s. Finally,

$$N' = T - S' = T - S = N.$$ 

The proof is complete. □
1 Implication of Jordan-Chevalley Decomposition for Matrices

Let $V$ be a finite dimensional vector space over a field $F$ and $T$ an operator on $V$. As in the previous lecture, we assume that the characteristic polynomial of $T$ is a product of linear factors in $F[x]$: \[ \Delta_T(x) = (x - \lambda_1)^{\mu_1}(x - \lambda_2)^{\mu_2} \cdots (x - \lambda_k)^{\mu_k} \]

By the results of the last lecture, now we know the following: Let $C_i = \ker((T - \lambda_i I)^{\mu_i})$. Then first of all, we showed that $V = C_1 \oplus C_2 \oplus \cdots \oplus C_k$.

The other important result, the Jordan-Chevalley decomposition theorem told us that there exist unique semisimple and nilpotent operators $S$ and $N$ such that $T = S + N$, and such that $T, S, N$ pairwise commute. The proof of that result shows that each $C_i$ above is an invariant subspace for all three operators $T, S, N$, and furthermore $S$ restricted to $C_i$ is simply equal to $\lambda_i I$.

We now want to understand the implications of these results for matrices representing the operator $T$ in suitable bases. If we select arbitrary bases $B_1, B_2, \ldots, B_k$ for the subspaces $C_1, C_2, \ldots, C_k$ respectively, then the matrix $A$ representing $T$ with respect to the basis $B = B_1 \cup \cdots \cup B_k$ is of the block diagonal form

\[
A = \begin{bmatrix}
A_1 & 0 & \cdots & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & A_{k-1} & 0 \\
0 & \cdots & \cdots & \cdots & A_k
\end{bmatrix}
\]

We can do better than this: So far, we selected arbitrary bases for $C_1, C_2, \ldots, C_k$. We can tidy things up by making more careful choices of bases. The following result is the first improvement that we can make:

**Proposition 1.** With the assumptions and notation above, there exists a basis for $C_i$ such that the matrix $A_i$ for $T$ restricted to $C_i$ is of the form

\[
A_i = \begin{bmatrix}
\lambda_i & * & \cdots & * \\
0 & \lambda_i & * & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \lambda_i \\
0 & \cdots & \cdots & 0
\end{bmatrix},
\]

(each * denotes an arbitrary element of the field $F$), the matrix for $S$ restricted to $C_i$ is $\lambda_i I$, and the matrix for $N$ restricted to $C_i$ is strictly upper triangular.

**Proof:** Recall that $S$ restricted to $C_i$ is equal to $\lambda_i I$ and $N$ restricted to $C_i$ is equal to $T - \lambda_i I$. Let $0 \neq v_1 \in C_i \cap \ker(T - \lambda_i)$. We clearly have $Tv_1 = \lambda_i v_1$ so that $v_1$ is an eigenvector for $T$ with eigenvalue $\lambda_i$. This will be our first basis vector. To find the second one, look at the quotient vector space $W_2 = C_i/\text{Span}(v_1)$. Since all transformations $T, S, N$ send $\text{Span}(v_1)$ to
itself, they induce linear operators on \( W_2 \). Furthermore, the operator that \( N \) induces on \( W_2 \) is still nilpotent. Pick \( v_2 \in C_i \) such that its equivalence class in \( W_2 \) is non-zero and in the kernel of \( N \). But then this means, there exists a constant \( a_{12} \in F \) such that
\[
Tv_2 = a_{12}v_1 + \lambda_i v_2.
\]
Let \( v_2 \) be the second basis vector. Its independence from \( v_1 \) is clear.

It is clear how we can proceed inductively: Having chosen \( v_1, \ldots, v_{m-1} \), define the quotient vector space \( W_m = C_i/\text{Span}(v_1, \ldots, v_{m-1}) \). If this vector space is trivial, this means that \( \{v_1, \ldots, v_{m-1}\} \) is a basis for \( C_i \), so we stop. If not, then \( N \) induces a nilpotent operator on it. So pick \( v_m \in C_i \) such that its equivalence class in \( W_m \) is non-zero and belongs to the kernel of \( N \). Then it is an eigenvector for \( T \) in this quotient space. Lifting back to \( C_i \), we see that
\[
Tv_m = a_{1m}v_1 + a_{2m}v_2 + \ldots + a_{m-1m}v_{m-1} + \lambda_i v_m
\]
for some \( a_{1m}, \ldots, a_{m-1m} \in F \).

Continuing in this way, at some stage we obtain a basis \( \{v_1, \ldots, v_m\} \) for \( C_i \). It is clear from the expression for each \( Tv_j \) that the matrix \( A_i \) representing \( T \) with respect to this basis will be upper triangular with diagonal elements \( \lambda_i \) (and the \( a_{jl} \) above are going to be the entries above the diagonal). Since \( S \) restricted to \( C_i \) is \( \lambda_i I \), it is clear that \( N = T - S \) will be represented by \( A_i - \lambda_i I \) which will be strictly upper triangular. This finishes the proof. \( \square \)

## 2 Jordan Form

Despite the nice structural result proved in the last section that shows that for every linear operator there is a choice of basis such that the matrix \( A \) representing \( T \) is of a special form, there is still some more room for standardization. We will show below that by an even more careful choice of basis, the representing matrix \( A \) can be chosen to be a matrix in so called Jordan form. We will first define a matrix in Jordan form, and then we will prove this result.

### Definition 1

Let \( F \) be a field. An \( m \times m \) matrix \( J_\lambda \) over \( F \) is said to be a **Jordan block** if it is of the form
\[
J_\lambda = \begin{bmatrix}
\lambda & 1 & 0 & \ldots & 0 \\
0 & \lambda & 1 & 0 & \ldots \\
0 & 0 & \lambda & 1 & \ldots \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \ldots & \lambda \\
0 & \ldots & \ldots & 0 & \lambda \\
\end{bmatrix}
\]
for some \( \lambda \in F \) (namely, \((J_\lambda)_{ii} = \lambda \) for all \( i = 1, \ldots, m \), \((J_\lambda)_{i+1} = 1 \) for all \( i = 1, \ldots, m-1 \) and all other entries are 0).

### Definition 2

Let \( F \) be a field. An \( n \times n \) matrix \( J \) over \( F \) is said to be a matrix in **Jordan form** (or a Jordan matrix) if it is in the block form
\[
J = \begin{bmatrix}
J_{\lambda_1} & 0 & \ldots & 0 \\
0 & J_{\lambda_2} & 0 & \ldots \\
\vdots & \vdots & \ddots & \ddots \\
0 & \ldots & 0 & J_{\lambda_{r-1}} \\
0 & \ldots & \ldots & 0 & J_{\lambda_r} \\
\end{bmatrix}
\]
where each \( J_{\lambda_i} \) is a Jordan block.
Remark: The \( \lambda_i \)'s don’t need to be distinct. The Jordan blocks can be of various sizes.

Example: The matrices

\[
A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\]

are both Jordan matrices. The matrix \( A \) has 3 Jordan blocks; the first is \( 2 \times 2 \) with \( \lambda_1 = 3 \), the second is \( 1 \times 1 \) with \( \lambda = 3 \) and the third is \( 1 \times 1 \) with \( \lambda = -2 \). The matrix \( B \) has 1 Jordan block, it is \( 3 \times 3 \) with \( \lambda = 0 \).

Example: Any diagonal matrix is in Jordan form, whose Jordan blocks are all \( 1 \times 1 \).

We now want to prove the main result of this lecture:

**Theorem 1.** Let \( V \) be a finite dimensional vector space over a field \( F \). Let \( T \) be a linear operator on \( V \) such that its characteristic polynomial is a product of linear factors in \( F[x] \). Then there exists a basis \( B \) of \( V \) such that the matrix representing \( T \) in this basis is in Jordan form.

**Proof:** With the notation above, it will be enough to show that we can find a basis for each subspace \( C_i \) such that the corresponding matrix \( A_i \) representing the restriction of \( T \) to \( C_i \) is in Jordan form. Since \( N \) restricted to \( C_i \) is nilpotent, there exists a minimal integer \( m \) such that \( N^m = 0 \) on \( C_i \). If \( m = 1 \), then \( N = 0 \) on \( C_i \) which means that \( T = \lambda_i I \) on \( C_i \), so it is in Jordan form (actually, diagonal). Suppose \( m \geq 2 \). Since \( m \) is minimal, \( N^{m-1} \neq 0 \) so there exists \( v_m \in C_i \) such that \( N^{m-1}v_m \neq 0 \). Let \( v_{m-1} = Nv_m, v_{m-2} = N^2v_m, \ldots, v_1 = N^{m-1}v_m \). Notice that for every \( j \) we have \( N^j v_j = 0 \) and \( N^{j-1}v_j \neq 0 \) by construction. We claim that these vectors are linearly independent. Indeed, suppose that

\[
c_1v_1 + \ldots + c_jv_j = 0
\]

where \( c_j \neq 0 \). Apply \( N^{j-1} \) to both sides of this equation. This implies that \( c_jN^{j-1}v_j = 0 \), which in turn says that \( c_j = 0 \), and this is a contradiction. Hence \( \{v_1, \ldots, v_m\} \) is linearly independent. Consider the action of \( N \) on the subspace \( \text{Span}(v_1, \ldots, v_m) \). Since \( Nv_1 = v_2, Nv_2 = v_3, \ldots, Nv_{m-1} = v_m \) we see that \( N \) is represented by the matrix

\[
\begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & 0 & 1 \\
0 & \ldots & \ldots & 0 & 0
\end{bmatrix}
\]

on the subspace \( \text{Span}(v_1, \ldots, v_m) \). Since \( T = \lambda_i I + N \), it is represented by a Jordan block on the same subspace.

Next, we want to pass to the quotient space \( C_i/\text{Span}(v_1, \ldots, v_m) \). If this space is trivial, then \( C_i = \text{Span}(v_1, \ldots, v_m) \) and there is nothing left to be proven. Suppose this space is not trivial. The operator \( N \) induces a nilpotent operator on \( C_i/\text{Span}(v_1, \ldots, v_m) \), say \( m' \) is the minimal integer such that \( N^{m'} = 0 \) on this space. Clearly \( m' \leq m \). Say that \( v \in C_i \) is a vector such that \( N^{m'-1}v \neq 0 \) where \( v \) denotes the equivalence class of \( v \) in the quotient space \( C_i/\text{Span}(v_1, \ldots, v_m) \). Lifting back to \( C_i \), the vector \( N^{m'}v \) must belong to \( \text{Span}(v_1, \ldots, v_m) \) so we must have

\[
N^{m'}v = a_1v_1 + a_2v_2 + \ldots + a_mv_m
\]
for some \( a_1, \ldots, a_m \in F \). We claim that \( a_m = a_{m-1} = \ldots = a_{m-m'+1} = 0 \). Suppose not, and say that \( j \geq m - m' + 1 \) is the greatest integer such that \( a_j \neq 0 \). Apply \( N^{j-1} \) to both sides of the equation above. This will give
\[
N^{m'+j-1}v = a_jN^{j-1}v_j \neq 0.
\]
But \( m' + j - 1 \geq m \) so \( N^m v \neq 0 \), a contradiction. So \( a_m = a_{m-1} = \ldots = a_{m-m'+1} = 0 \). Next, define the vector
\[
w_{m'} = v - a_{m-m'}v_m - a_{m-m'-1}v_{m-1} - \ldots - a_1v_{m'+1}.
\]
It is clear that \( \overline{w_{m'}} = \overline{v} \) in \( C_i/Span(v_1, \ldots, v_m) \). Also, by direct computation we have \( N^{m'}w_{m'} = 0 \). Now, set
\[
w_{m'-1} = Nw_{m'}, \ w_{m'-2} = Nw_{m'-1}, \ldots, w_1 = Nw_2.
\]
Then \( Span(w_1, \ldots, w_{m'}) \) is a \( T \)-invariant subspace and on this subspace \( T \) is represented by a Jordan block with respect to the basis \( \{w_1, \ldots, w_{m'}\} \), just like in the previous paragraph.

One proceeds inductively in the same manner by passing to \( C_i/Span(v_1, \ldots, v_m, w_1, \ldots, w_{m'}) \). We leave out the details. \( \square \)
Computing the Jordan Form of a Matrix

Let $T$ be a linear operator on a finite dimensional vector space $V$ over a field $F$. Suppose that the characteristic polynomial of $T$ is a product of linear polynomials in $F[x]$. We proved in the last lecture that there exists a basis for $V$ such that the matrix representing $T$ with respect to this basis is in Jordan form. This result has an immediate corollary for matrices:

**Corollary 1.** Let $A \in M_{n \times n}(F)$ be a matrix whose characteristic polynomial is a product of linear polynomials in $F[x]$. Then there exists a matrix $J$ in Jordan form and an invertible matrix $P$ such that

$$A = PJP^{-1}$$

**Definition 1.** The matrix $J$ in the above corollary is often referred to as the **Jordan canonical form** of the matrix $A$.

These results tell us in a sense what we can do in the cases when $A$ (or $T$) is not diagonalizable, therefore generalize our previous discussion. One important comment is that the condition for the characteristic polynomial to be a product of linear factors is automatically satisfied in an algebraically closed field, for instance when $F = \mathbb{C}$.

One question that we did not address is the uniqueness of Jordan form: Uniqueness actually does hold, except for an obvious possibility of changing the places of Jordan blocks. We will skip the proof. There are no further technical obstacles to prove it though; it can be left as a challenging exercise at this point.

The proof of existence of Jordan form was somewhat opaque, in the sense that it is tricky to follow the details for an actual computation. Here, we will solve several examples in order to see how Jordan canonical forms can actually be computed. Instead of giving a full algorithm that covers all possibilities (which is possible but involves some elaborate combinatorics) we will try to select enough examples which will allow the reader to guess the pattern to a certain extent.

**Example:** Find the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix}$$

in $M_{2 \times 2}(\mathbb{R})$.

Let us find the characteristic polynomial of $A$ first:

$$\Delta_A(x) = \det(xI - A) = x^2 - 4x + 4 = (x - 2)^2.$$ 

Therefore the characteristic polynomial is a product of linear factors in $\mathbb{R}[x]$, so the matrix has a real Jordan canonical form. We see that the only eigenvalue is $\lambda = 2$. At this point, without any further calculation, there are only two possibilities for the Jordan form of a $2 \times 2$ matrix with only a single eigenvalue $\lambda = 2$:

$$J_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$
Which of these possibilities is actually the Jordan form of \( A \)? We claim that it must be \( J_2 \).

Indeed, suppose that it is \( J_1 \). But we see that \( J_1 = 2I \). So if \( A = PJ_1P^{-1} \), then \( A = P(2I)P^{-1} = 2I \). But \( A \neq 2I \), contradiction. Therefore this cannot be the case.

At this point, we decided that the Jordan form of \( A \) is \( J_2 \). But this doesn’t yet give us a matrix \( P \) such that \( A = PJ_2P^{-1} \). To find such a \( P \), suppose that

\[
P = \begin{bmatrix} v_1 & v_2 \end{bmatrix}
\]

where \( v_1 \) and \( v_2 \) are column vectors of \( P \). Let us look at the equation \( AP = PJ_2 \) in detail:

\[
AP = PJ_2
\]

\[
A[v_1 v_2] = [v_1 v_2] \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}
\]

\[
[Av_1 \mid Av_2] = [2v_1 \mid v_1 + 2v_2]
\]

Therefore \( v_1 \) and \( v_2 \) must be solutions of the two linear systems

\[
Av_1 = 2v_1, \quad Av_2 = v_1 + 2v_2
\]

The first equation tells us that \( v_1 \) must be an eigenvector of \( A \). The second vector \( v_2 \) is not an eigenvector, but since the equation is resembling an eigenvector equation, sometimes such a vector is called a generalized eigenvector.

Find \( v_1 \) by solving \((A - 2I)v_1 = 0\):

\[
\begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} -2k \\ k \end{bmatrix}, k \in \mathbb{R}
\]

Select a non-zero value of \( k \) in order to get a first column for \( P \) (it must be non-zero since \( P \) needs to be invertible). For instance choose \( k = 1 \) so that

\[
v_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}
\]

Next, find \( v_2 \) by solving \((A - 2I)v_2 = v_1\):

\[
\begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \ell \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \ell \in \mathbb{R}.
\]

Select a value of \( \ell \) in order to get a second column for \( P \). This time, \( \ell \) does not have to be nonzero since the resulting matrix \( P \) will be invertible regardless of the choice of \( \ell \) (please check this). Select for instance, \( \ell = 0 \) (any other choice would be equally valid). So we get

\[
v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

Now we have the matrix \( P \):

\[
P = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}
\]

Finally, the equality \( A = PJ_2P^{-1} \) looks like

\[
\begin{bmatrix} 0 & -4 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}^{-1}
\]
We remark that the Jordan canonical form $J_2$ is unique, although the matrix $P$ is far from being unique; we had certain choices for $k$ and $\ell$ along the way.

**Example:** Find the Jordan canonical form of the matrix

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

in $M_{3 \times 3}(\mathbb{F}_3)$ where $\mathbb{F}_3$ is the finite field with 3 elements. Let us first find the characteristic polynomial of $A$.

$$\Delta_A(x) = \begin{vmatrix} x-2 & 2 & 1 \\ 0 & x & 1 \\ 0 & 2 & x-1 \end{vmatrix} = (x-2)^3.$$ (Remember that we are working over the field $\mathbb{F}_3$.) There are three possibilities for the Jordan form (up to the ordering of Jordan blocks):

$$J_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Which of these is actually the Jordan form? Let us try to find the minimal polynomial $\delta_A(x)$ of $A$. We know that the minimal polynomial must be one of the three polynomials

$$(x-2), \quad (x-2)^2, \quad (x-2)^3.$$ 

Similar matrices have the same minimal polynomial, therefore $A$ and its Jordan canonical form should have the same minimal polynomial. This should give us a clue. Now, first of all it is clear that $A - 2I \neq 0$ therefore $\delta_A(x) \neq (x-2)$. Let us try $(x-2)^2$:

$$(A - 2I)^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

therefore we deduce that $\delta_A(x) = (x-2)^2$. It can be quickly checked that the minimal polynomial of $J_1$ is $(x-2)$, the minimal polynomial of $J_2$ is $(x-2)^2$ and the minimal polynomial of $J_3$ is $(x-2)^3$. We deduce that the Jordan canonical form of $A$ is $J_2$.

Let us now try to find an invertible matrix $P$ such that $AP = PJ_2$. Say $P = [v_1|v_2|v_3]$ where $v_1, v_2, v_3$ are the columns of $P$. Writing the equation $AP = PJ_2$, we get

$$AP = PJ_2$$

$$A [v_1|v_2|v_3] = [v_1|v_2|v_3] \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[Av_1|Av_2|Av_3] = [2v_1 + v_2|2v_2|2v_3]$$

Therefore, $v_1, v_2, v_3$ must be the solutions of

$$Av_1 = 2v_1, \quad Av_2 = v_1 + 2v_2, \quad Av_3 = 2v_3$$
So, $v_1$ and $v_3$ must be eigenvectors of $A$ and $v_2$ must be a generalized eigenvector. The eigenvector equation $(A - 2I)v = 0$ gives us

$$
\begin{bmatrix}
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2 \\
v_3
\end{bmatrix}
= 0
\Rightarrow
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
= s
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
+ t
\begin{bmatrix}
0 \\
1 \\
1
\end{bmatrix}, s, t \in \mathbb{F}_3
$$

where $s$ and $t$ are not both 0. We have to be careful now: In order to find $v_2$, we would like to solve the equation $(A - 2I)v_2 = v_1$. But we have some freedom for choosing $v_1$ and it is unclear which choices would lead to a solution for $v_2$ and not leave us with an inconsistent linear system.

So, at this point, let us write the form of the most general eigenvector in place of $v_1$:

$$
\begin{bmatrix}
0 & 1 & 2 \\
0 & 1 & 2 \\
0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
v_2 \\
v_2 \\
v_2
\end{bmatrix}
= \begin{bmatrix}
s \\
t \\
t
\end{bmatrix}.
$$

It is easy to see that this system has a solution if and only if $s = t$, therefore we must select the eigenvector $v_1$ accordingly. Set

$$
v_1 = \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
$$

Then select a solution $v_2$ of the linear system above. For instance, we can take

$$
v_2 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
$$

What about $v_3$? All that we want is it should be an eigenvector and \{v_1, v_3\} should be a basis for the eigenspace $W_2$. There are many choices. Take for instance by setting $s = 1$ and $t = 0$:

$$
v_3 = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
$$

This now gives us the matrix $P$. We have

$$
P = \begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
$$

The equation $A = PJP^{-1}$, explicitly written, looks like

$$
A = \begin{bmatrix}
1 & 0 & 1 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 1 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}^{-1}
$$

**Example:** Find the Jordan canonical form of the following matrix in $M_{6 \times 6}(\mathbb{R})$:

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$
Let us start by finding its characteristic polynomial:

\[
\Delta_A(x) = \begin{vmatrix}
    x - 1 & 0 & 0 & 0 & -1 & 0 \\
    0 & x - 1 & -1 & -1 & 0 & 1 \\
    0 & 0 & x - 1 & 0 & 0 & 0 \\
    0 & 0 & 0 & x - 1 & 0 & 0 \\
    0 & 0 & -1 & 0 & x - 1 & 1 \\
    0 & 0 & 0 & 0 & 0 & x - 1
\end{vmatrix} = (x - 1)^6
\]

(easy way to compute this determinant: Observe that only the identity permutation contributes a non-zero term.) Hence we have a six-fold repeated eigenvalue, \( \lambda = 1 \). We have quite a lot of possibilities for the Jordan form (up to ordering of Jordan blocks):

\[
J_1 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_2 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_3 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_4 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_5 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_6 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_7 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_8 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_9 = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_{10} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
J_{11} = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

We notice that there is a bijection between the possible Jordan forms for a given eigenvalue and all integer partitions of the matrix size. In particular, the matrices above correspond to the partitions of the number 6 in the following order: 1+1+1+1+1+1, 2+1+1+1+1+1, 2+2+1, 3+1+1+1, 3+2+1, 3+3, 4+1+1, 4+2, 5+1, 6.

Which of these is the actual Jordan canonical form? Let us try to find the minimal polynomial \( \delta_A(x) \). It must be one of the polynomials

\((x - 1),
(x - 1)^2, (x - 1)^3, (x - 1)^4, (x - 1)^5, (x - 1)^6\)
It is certainly not \( x - 1 \), since otherwise we would have \( A = I \), but this is not true.

\[
(A - I)^2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}^2 = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

In particular \( (A - I)^2 \neq 0 \), hence the minimal polynomial of \( A \) cannot be \( (x - 1)^2 \). But by one more similar computation we immediately see that \( (A - I)^3 = 0 \). Therefore, \( \delta_A(x) = (x - 1)^3 \).

Let us now check which \( J \)'s have this minimal polynomial, knowing that \( A \) and its Jordan canonical form must have the same minimal polynomial. The minimal polynomial of \( J_1 \) is \( x - 1 \), the minimal polynomials of \( J_2, J_3 \) and \( J_4 \) are \( (x - 1)^2 \), the minimal polynomial of \( J_5, J_6 \) and \( J_7 \) are \( (x - 1)^3 \), the minimal polynomials of \( J_8, J_9 \) and \( J_{10} \) are \( (x - 1)^4 \), the minimal polynomial of \( J_{11} \) is \( (x - 1)^5 \) and the minimal polynomial of \( J_{12} \) is \( (x - 1)^6 \). One can compute all of these directly, but the general rule is that the power of \( (x - 1) \) in the minimal polynomial will be equal to the the largest integer in the partition corresponding to the Jordan form. This is not difficult to generalize and prove.

We deduce that the Jordan canonical form of \( A \) must be one of \( J_5, J_6 \) or \( J_7 \). How can we distinguish between these three forms? Notice that if \( J \) is the Jordan form of \( A \), then the matrix \( J - I \) will be similar to the matrix \( A - I \), therefore the vector spaces \( \ker(J - I) \) and \( \ker(A - I) \) will have the same dimension, namely this tells us that we could distinguish between these possibilities by looking at the dimension of the eigenspace for \( \lambda = 1 \). Looking at the equation \( (A - I)v = 0 \), we get

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

The coefficient matrix has rank 3, hence it also has nullity equal to 3 by the rank-nullity theorem. Therefore the eigenspace \( W_1 \) has dimension 3. Now, observe directly that

\[
\text{rank}(J_5 - I) = 2, \quad \text{rank}(J_6 - I) = 3, \quad \text{rank}(J_7 - I) = 4.
\]

By using the rank-nullity theorem for each of these matrices, we get

\[
\text{nullity}(J_5 - I) = 4, \quad \text{nullity}(J_6 - I) = 3, \quad \text{nullity}(J_7 - I) = 2.
\]

Since the correct number is given only by \( J_6 \), we deduce that the Jordan canonical form of \( A \) must be \( J_6 \).

The diligent reader will be probably worried about the following possibility: What if the matrices were larger (or there were more eigenvalues) and there were still more possibilities remaining after comparing these dimensions? One can continue as follows: Look at the dimensions of \( \ker((A - \lambda I)^2) \), \( \ker((A - \lambda I)^3) \), ... for each eigenvalue until only one candidate remains.
It can be proven that these dimensions eventually characterize the whole partition and we can fully retrieve the Jordan form.

Let us also find an invertible matrix $P$ such that $AP = PJ_6$. Let us write $P$ in terms of its columns: $P = [v_1|v_2|v_3|v_4|v_5|v_6]$. Then,

$$AP = PJ_6$$

$$A[v_1|v_2|v_3|v_4|v_5|v_6] = [v_1|v_2|v_3|v_4|v_5|v_6]$$

$$[Av_1|Av_2|Av_3|Av_4|Av_5|Av_6] = [v_1|v_1 + v_2|v_2 + v_3|v_4|v_4 + v_5|v_6]$$

Therefore, we have the following linear systems:

$$Av_1 = v_1, \quad Av_2 = v_1 + v_2, \quad Av_3 = v_2 + v_3, \quad Av_4 = v_4, \quad Av_5 = v_4 + v_5, \quad Av_6 = v_6$$

Therefore, $v_1, v_4$ and $v_6$ will be eigenvectors, the others will be certain generalized eigenvectors.

Let us find all eigenvectors of $A$ by solving $(A - I)v = 0$:

$$\begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} v = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} \Rightarrow v = r \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{bmatrix} + t \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}$$

Now, in order to find $v_2$ and $v_5$ we need to solve the equation $(A - I)w = v$ where $v$ is an eigenvector. This linear system looks like

$$\begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} w = \begin{bmatrix}
r \\
s \\
t \\
0 \\
0 \\
t
\end{bmatrix}$$

This system is consistent if and only if $t = 0$, and its solutions are of the form

$$w = \begin{bmatrix}
a \\
b \\
c \\
s \\
r \\
c
\end{bmatrix}.$$
Finally, in order to find $v_3$ we need to solve $(A - I)u = w$ where $w$ is as above. This reads

$$\begin{bmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} u = \begin{bmatrix}
a \\
b \\
c \\
s \\
r \\
c
\end{bmatrix}.$$  

This system is consistent if and only if $c = s = 0$.

To sum up: The “deepest” vector is $v_3$. In order to make all steps along the way solvable, we need to select the eigenvector $v_1$ such that $s = t = 0$. So, for instance we can take $r = 1, s = t = 0$ and get

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$  

The vector $v_2$ must satisfy $(A - I)v_2 = v_1$ and furthermore we must have $c = 0$ in the resulting solution. One solution is (by taking $a = b = c = 0$)

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$  

Then, we find $v_3$ by solving $(A - I)v_3 = v_2$. One solution is

$$v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$  

Next, let us select $v_4$. The restrictions on it are: It must be an eigenvector independent of $v_1$, and $t = 0$ must be satisfied so that later we can solve for $v_5$. So we could take $r = 0, s = 1$ and $t = 0$ in order to get

$$v_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$
Next, $v_5$ must be a solution of $(A - I)v_5 = v_4$. One solution is (by taking $a = b = c = 0$)

$$v_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Finally, $v_6$ must be an eigenvector independent of $v_1, v_4$. A simple way to get it is to set $r = s = 0$ and $t = 1$:

$$v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

At this point, we are ready to write $P$ and finish the problem:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
MATH 262

Practice Problems for Quiz 5

The quiz 5 topics for MATH 262 are Quadratic Forms, Principal Axis Theorem, Sylvester’s Law of Inertia, Singular Value Decomposition, Polar Decomposition.

Practice Problems:

1. Consider the two variable quadratic form \( q(x, y) = x^2 + 2xy + 6y^2 \) over the field \( \mathbb{F}_p \), the finite field with \( p \) elements, where \( p \) is an odd prime. Find a diagonal quadratic form equivalent to \( q \). For which values of \( p \) does the quadratic form \( q \) have rank 2?

2. Find an orthogonal diagonalization of the real quadratic form \( q(x, y) = 4x^2 + 8xy + 2y^2 \). What is the rank and signature of this form?

3. Write down 15 real quadratic forms in 4 variables such that no two of them are equivalent to one another. Can you generalize to \( \frac{(n+1)(n+2)}{2} \) non-equivalent real quadratic forms in \( n \) variables?

4. Consider the zero locus of the equation \( x^2 - 10xy + 3y^2 = 8 \) in the two dimensional real affine plane. Find a corresponding quadratic form in 3 variables and find an orthogonal diagonalization of it. What is its signature? Is this zero locus an ellipse, a hyperbola or some other figure?

5. Consider the zero locus of the equation \( x^2 - 2y^2 + z^2 - 2xy + 2yz + 4zx = 27 \) in the three dimensional real affine space. Find a corresponding quadratic form in 4 variables and find an orthogonal diagonalization of it. What is its signature? Is this zero locus an ellipsoid, a hyperboloid of one sheet, a hyperboloid of two sheets or some other figure?

6. Let \( A \) be an \( n \times n \) real or complex matrix. Prove that \( A \) is invertible if and only if 0 is not a singular value of \( A \). (Note: The \( \sigma_i \)'s appearing in the singular value decomposition of \( A \) (which are non-negative real numbers) are called the singular values of \( A \)).

7. Find the singular value decompositions and pseudoinverses of each of the following matrices:
\[
A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\]

8. Let \( A \) be an \( n \times n \) real or complex matrix. Let \( \sigma_{\text{min}} \) denote the smallest and \( \sigma_{\text{max}} \) denote the largest singular value of \( A \). Prove that if \( \lambda \) is an eigenvalue of \( A \), then \( \sigma_{\text{min}} \leq |\lambda| \leq \sigma_{\text{max}} \).

9. Show that the matrix \( A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \) is positive definite. Find a positive definite matrix \( B \) such that \( B^4 = A \).

10. Find the polar decompositions of the following \( 2 \times 2 \) matrices:
\[
A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3/5 & 4i/5 \\ 4i/5 & 3/5 \end{bmatrix}
\]
MATH 262 Practice Problems for Quiz 6

Below are some practice problems for Quiz 6. The topics covered are: Jordan-Chevalley Decomposition. Jordan Form. Computing the Jordan Form of an Operator.

1. Suppose that \( V \) is a finite dimensional vector space and \( T \) a linear operator on \( V \). Show that if \( \ker(T) = \ker(T^2) \) then \( \ker(T) = \ker(T^m) \) for every integer \( m \geq 1 \).

2. Let \( A \) be the \( 4 \times 4 \) matrix over \( \mathbb{F}_2 \) whose entries are all 1’s. Find the Jordan form \( J \) of \( A \) and an invertible matrix \( P \) such that \( A = PJP^{-1} \).

3. Find all \( 5 \times 5 \) nilpotent Jordan matrices, up to an ordering of the Jordan blocks. How many of them are there?

4. Find all possible Jordan matrices whose characteristic polynomial is \( x^3(x - 1)^4 \) and whose minimal polynomial is \( x^2(x - 1)^2 \).

5. Classify all \( 3 \times 3 \) complex matrices \( A \) satisfying \( A^3 = I \) up to similarity.

6. Find the Jordan form \( J \) of the matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

over \( \mathbb{R} \). Find an invertible matrix \( P \) such that \( A = PJP^{-1} \).

7. Find the Jordan form \( J \) of the matrix

\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 & 0 & 0 \\
1 & 1 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\]

over \( \mathbb{R} \). Find an invertible matrix \( P \) such that \( A = PJP^{-1} \).

8. Find the Jordan form \( J \) of the matrix

\[
A = \begin{bmatrix}
1 & 2 & -1 \\
0 & 2 & 0 \\
1 & -2 & 3
\end{bmatrix}
\]

over \( \mathbb{R} \). Find an invertible matrix \( P \) such that \( A = PJP^{-1} \).

9. The characteristic polynomial of the matrix

\[
A = \begin{bmatrix}
7 & 1 & 2 & 2 \\
1 & 4 & -1 & -1 \\
-2 & 1 & 5 & -1 \\
1 & 1 & 2 & 8
\end{bmatrix}
\]

over \( \mathbb{R} \) is \( (x - 6)^4 \). Using this, find the Jordan form \( J \) of \( A \). Find an invertible matrix \( P \) such that \( A = PJP^{-1} \).
10. Find the Jordan form $J$ of the matrix

$$A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{bmatrix}$$

over $\mathbb{R}$. Find an invertible matrix $P$ such that $A = PJP^{-1}$. 