Non-Homogeneous Wave Equation (Waves with a Source)

We'll consider:

1) Homogeneous Wave Eqn. with Non-Zero Initial Conditions
   \( u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = f(x), \quad u_t(x,0) = g(x) \)

2) Non-Homogeneous Wave Eqn. with Zero Initial Conditions
   \( u_{tt} - c^2 u_{xx} = h(x,t), \quad w(x,0) = 0, \quad w_t(x,0) = 0 \)

Here \( h(x,t) \) can be considered as an external force acting on an infinite string. Since our equation is linear, we can write the general solution as a sum of two solutions coming from two problems in ODE.

Recall: For any \( P, Q \) (cont'd) differentiable on \( \mathbb{R}^2 \) by Green's

Proposition: \( \int \int \int (P_t dx + Q_t dy + R_t dz) \) by Green's
Now let's integrate $W_{tt} - c^2 W_{xx} = h(x,t)$ on $\Delta$

\[
\iint h \, dx \, dt = \iint \left( W_{tt} - c^2 W_{xx} \right) \, dx \, dt
\]

We can use Green's thm to left hand side

\[
\iint (W_{tt} - c^2 W_{xx}) \, dx \, dt = \int \int (-c^2 W_x \, dt - W_t \, dx)
\]

We'll evaluate each piece:

1) On $L_0$, ie $(x,t) = (x,0)$, $dt = 0$, $W_t(x,0) = 0$

\[
\int_{L_0} -c^2 W_x \, dt - W_t \, dx = \int_{x_0 \to c t_0} 0 \, dx = 0
\]

2) On $L_1$, $x+ct = x_0 + ct_0$, so $dx + c \, dt = 0$, ie $dt = -\frac{1}{c} \, dx$

\[
\int_{L_1} -c^2 W_x \, dt - W_t \, dx = \int_{L_1} -c^2 W_x \left( -\frac{1}{c} \, dx \right) - W_t \left( -\frac{1}{c} \right) \, dt = \int_{L_1} W_x \, dx + W_t \, dt = C \int W \, dx
\]

\[
= C \left( W(x_0,t_0) - W(x_0 + ct_0,0) \right) = C \cdot W(x_0,t_0)
\]

By Initial Cond.

3) On $L_2$, $x-ct = x_0 - ct_0$, $dx - c \, dt = 0$, $ie \, dt = \frac{1}{c} \, dx$. Similar to (2) we get

\[
\int_{L_2} -c^2 W_x \, dt - W_t \, dx = -C \int W \, dx = -C \left( W(x_0 - ct_0) - W(x_0, t_0) \right) = C \cdot W(x_0, t_0)
\]

By Initial Cond.

By adding these three integrals:

\[
\iint h \, dx \, dt = 0 + 2C \cdot W(x_0,t_0)
\]

Hence,

\[
W(x_0,t_0) = \frac{1}{2C} \iint h \, dx \, dt
\]
Example: Determine the solution of

\[ u_{tt} - u_{xx} = 1 \]

\[ u(x,0) = \cos x \]

\[ u_t(x,0) = x \]

**Soln:** Here \( c = 1 \), so the characteristic curves are \( x + t = x_0 + t_0 \) and \( x - t = x_0 - t_0 \). If we consider \( U = V + W \) as in the proof,

\[ V(x,t) = \frac{\cos(x-t) + \cos(x+t)}{2} + \frac{1}{2.1} \int_{x-t}^{x+t} \right. \]

\[ \left. \frac{(x+t)^2 - (x-t)^2}{2} \right) \]

\[ = \frac{\cos(x-t) + \cos(x+t)}{2} + \frac{1}{4} \cdot 2x \cdot 2t \]

\[ W(x,t) = \frac{1}{2.1} \int_{x-t}^{x+t} \int_i 1 \cdot dA = \frac{1}{2} \cdot \frac{(x+t-x+t)t}{2} \]

\[ = \frac{t^2}{2} \]

\[ U(x,t) = \frac{\cos(x-t) + \cos(x+t)}{2} + x \cdot t + \frac{t^2}{2} \]

**Example:**

\[ u_{tt} - 4u_{xx} = e^x \]

\[ u(x,0) = x \]

\[ u_t(x,0) = 1 \]  [Initial Conditions]

\[ u(0,t) = 0 \]  [Boundary Condition]

**Soln:** Again, we can separate it into two problems for \( V, W \) such that \( U = V + W \).
First, \[ V_{tt} - 4V_{xx} = 0 \quad W_{tt} - 4W_{xx} = e^x \]
\[ V(x,0) = x \quad W(x,0) = 0 \]
\[ V_t(x,0) = 1 \quad W_t(x,0) = 0 \]
\[ V(0,t) = 0 \quad W(0,t) = 0 \]

It's the semi-infinite string with a fixed end, so as we did before:

\[ V(x,t) = \begin{cases} 
\frac{x+2t + x-2t}{2} + \frac{1}{2.2} \int_{x-2t}^{x+2t} \frac{1}{s} \, ds & \text{if } x > 2t \\
\frac{(x+2t) - (2t-x)}{2} + \frac{1}{2.2} \int_{2t-x}^{x+2t} \frac{1}{s} \, ds & \text{if } x < 2t 
\end{cases} \]

Similarly, when \( x > 2t \), \[ W(x,t) = \frac{1}{2.2} \int_{x-2t}^{x+2t} e^x \, dA \]

But, when \( x < 2t \)

Note that when \((x,t) = (0,t)\), then \(D\) will be just a line segment and the double integral is equal to zero. Hence, \( W(0,t) = 0 \) is automatically satisfied.

Let me compute \( x > 2t \) case and leave \( x < 2t \) to you.

\[ W(x,t) = \frac{1}{4} \int_{0}^{x+2t-2t} \int_{x-2t+2t}^{x+2t} e^x \, dx \, dt = \frac{1}{4} \int_{0}^{t} \left( e^{x+2t-2t} - e^{x-2t+2t} \right) dt \]

\[ = \frac{e^{x+2t}}{4} \left( \frac{e^{2t}}{2} - \frac{e^{-2t}}{2} \right) \bigg|_{0}^{t} = -\frac{e^{x+2t}}{4} \cdot \frac{1}{2} \left( e^{2t} + e^{-2t} + 1 \right) \]
Now, we'll show that non-homogeneous wave equation with zero initial conditions is well-posed.

\[ u_{tt} - c^2 u_{xx} = h(x,t) \quad u(x,0) = 0 = u_t(x,0) \]

1) Existence: \[ u(x,t) = \frac{1}{2c} \int \int_{\Delta} h(x,t) \, dA \]

2) Uniqueness: Let \( U_1, U_2 \) be two solutions. Then, \( V = U_1 - U_2 \) satisfies \( v_{tt} - c^2 v_{xx} = 0, \quad V(x,0) = V_t(x,0) = 0 \) which has a unique solution as proved earlier. Obviously, that unique solution is \( V(x,t) = 0 \) which proves \( U_1 = U_2 \).

3) Stability: We'll prove that if \( h(x,t) \) is small, then \( u(x,t) \) is small for \( t \in [0,T] \) which is equivalent to the statement that if \( h(x,t) \) changes little, then \( u(x,t) \) changes little, too.

Now,

\[ |u(x,t)| = \frac{1}{2c} \int \int_{\Delta} |h(x,t)| \, dA \leq \frac{1}{2c} \int \int_{\Delta} |h(x,t)| \, dA \]

\[ \forall \varepsilon > 0, \text{ choose } \delta < \frac{2\varepsilon}{T^2} \text{ such that if } \]

\[ \sup_{t \in [0,T]} |h(x,t)| < \delta, \quad \text{then} \]

\[ \sup_{t \in [0,T]} |u(x,t)| < \varepsilon \]

\[ |u(x,t)| \leq \frac{1}{2c} \int \int_{\Delta} |h(x,t)| \, dA \leq \frac{1}{2c} \int \int_{\Delta} \delta \, dA = \frac{\delta}{2c} \text{ Area}(\Delta) \]

\[ = \frac{\delta}{2c} \left( x+ct - (x-ct) \right) \cdot t = \frac{\delta}{2c} ct^2 \leq \frac{\delta}{2c} cT^2 = \varepsilon \]

Therefore,

\[ \sup_{t \in [0,T]} |u(x,t)| < \varepsilon \]