LECTURE NOTES 9

HYPERBOLIC EQUATIONS
Part II
18 May-1 June 2020
Hyperbolic Equations

Hyperbolic equations put less stringent constraints on explicit methods. In this part, the stability of finite difference methods is explored in the context of a representative hyperbolic equation called the wave equation. The CFL condition will be introduced, which is, in general, a necessary condition for stability of the PDE solver.

Mesh for the Finite Difference Method. The filled circles represent known initial and boundary conditions. The open circles represent unknown values that must be determined.
Consider the partial differential equation

\[ u_{tt} = c^2 u_{xx} \]

for \( a \leq x \leq b \) and \( t \geq 0 \). We have \( B^2 - 4AC = 4C^2 \geq 0 \), so the equation is hyperbolic. This example is called the \textit{wave equation} with wave speed \( c \). Typical initial and boundary conditions needed to specify a unique solution are

\[
\begin{align*}
   u(x, 0) &= f(x) \text{ for all } a \leq x \leq b \\
   u_t(x, 0) &= g(x) \text{ for all } a \leq x \leq b \\
   u(a, t) &= l(t) \text{ for all } t \geq 0 \\
   u(b, t) &= r(t) \text{ for all } t \geq 0
\end{align*}
\]

Compared with the heat equation, for the wave equation, extra initial data are needed due to the higher order time derivative in the equation. Indeed, the wave equation describes the time evolution of a wave propagating along the \( x \)-direction. To specify what happens, we need to know the initial shape of the wave and the initial velocity of the wave at each point. The wave equation models a wide variety of phenomena, from magnetic waves in the sun's atmosphere to the oscillation of a violin string. The equation involves an amplitude \( u \), which for the violin represents the physical displacement of the string. For a sound wave traveling in air, \( u \) represents the local air pressure.
We will apply the Finite Difference Method to the wave equation and analyze its stability. The FDM operates on a grid (as shown in the 1st page). The grid points are \((x_i, t_j)\), where \(x_i = a + ih\), \(t_j = jk\), for step sizes \(h\) and \(k\). Let us represent the approximation to the solution \(u(x_i, t_j)\) by \(w_{ij}\). To discretize the wave equation, the second partial derivatives are replaced by the centered-difference formula in both the \(x\) and \(t\) directions:

\[
\frac{w_{i,j+1} - 2w_{ij} + w_{i,j-1}}{k^2} - \frac{c^2 w_{i-1,j} - 2w_{ij} + w_{i+1,j}}{h^2} = 0.
\]

Setting \(\sigma = ck/h\), we can solve for the solution at the next time step and write the discretized equation as

\[
w_{i,j+1} = (2 - 2\sigma^2)w_{ij} + \sigma^2 w_{i-1,j} + \sigma^2 w_{i+1,j} - w_{i,j-1}.
\]

The formula (*) cannot be used for the first time step, since values at two prior times, \(j-1\) and \(j\), are needed. To solve the problem, we can introduce the three-point centered-difference formula to approximate the first time derivative of the solution \(u\):
Substituting initial data at the first time step \((x_i, t_1)\) yields
\[
g(x_i) = u_t(x_i, t_0) \approx \frac{w_{i1} - w_{i,-1}}{2k},
\]
which is equivalent to
\[
w_{i,-1} \approx w_{i1} - 2kg(x_i).
\]

Next substituting (***) in (*) we get
\[
w_{i1} = (2 - 2\sigma^2)w_{i0} + \sigma^2 w_{i-1,0} - w_{i1} + 2kg(x_i),
\]
which can be solved for \(w_{i1}\)
\[
w_{i1} = (1 - \sigma^2)w_{i0} + kg(x_i) + \frac{\sigma^2}{2}(w_{i-1,0} + w_{i+1,0}).
\]
Formula (***) is used for the first time step. This is the way the initial velocity information \( g \) enters the calculation. For all later time steps, formula (*) is used. Since second order formulas have been used for both space and time derivatives, the error of this Finite Difference Method will be \( O(h^2) + O(k^2) \).

To write the Finite Difference Method in matrix terms, define

\[
A = \begin{bmatrix}
2 - 2\sigma^2 & \sigma^2 & 0 & \cdots & 0 \\
\sigma^2 & 2 - 2\sigma^2 & \sigma^2 & \ddots & \vdots \\
0 & \sigma^2 & 2 - 2\sigma^2 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \sigma^2 \\
0 & \cdots & 0 & \sigma^2 & 2 - 2\sigma^2
\end{bmatrix}.
\]

The initial equation (***) can be written as
and the subsequent steps of (*) are given by

\[
\begin{bmatrix}
  w_{11} \\
  \vdots \\
  w_{m1}
\end{bmatrix}
= \frac{1}{2} A
\begin{bmatrix}
  w_{10} \\
  \vdots \\
  w_{m0}
\end{bmatrix}
+ k
\begin{bmatrix}
  g(x_1) \\
  \vdots \\
  g(x_m)
\end{bmatrix}
+ \frac{1}{2} \sigma^2
\begin{bmatrix}
  w_{00} \\
  0 \\
  \vdots \\
  0 \\
  w_{m+1,0}
\end{bmatrix},
\]

\[
\begin{bmatrix}
  w_{1,j+1} \\
  \vdots \\
  w_{m,j+1}
\end{bmatrix}
= A
\begin{bmatrix}
  w_{1j} \\
  \vdots \\
  w_{mj}
\end{bmatrix}
- \begin{bmatrix}
  w_{1,j-1} \\
  \vdots \\
  w_{m,j-1}
\end{bmatrix}
+ \sigma^2
\begin{bmatrix}
  w_{0j} \\
  0 \\
  \vdots \\
  0 \\
  w_{m+1,j}
\end{bmatrix}.
\]
Inserting the rest of the extra data, the two equations are written

\[
    \begin{bmatrix}
        w_{11} \\
        \vdots \\
        w_{m1}
    \end{bmatrix}
    = \frac{1}{2} A
    \begin{bmatrix}
        f(x_1) \\
        \vdots \\
        f(x_m)
    \end{bmatrix}
    + k
    \begin{bmatrix}
        g(x_1) \\
        \vdots \\
        g(x_m)
    \end{bmatrix}
    + \frac{1}{2}\sigma^2
    \begin{bmatrix}
        l(t_0) \\
        0 \\
        \vdots \\
        0 \\
        r(t_0)
    \end{bmatrix},
\]

and the subsequent steps of (*) are given by

\[
    \begin{bmatrix}
        w_{1,j+1} \\
        \vdots \\
        w_{m,j+1}
    \end{bmatrix}
    = A
    \begin{bmatrix}
        w_{1j} \\
        \vdots \\
        w_{mj}
    \end{bmatrix}
    - \begin{bmatrix}
        w_{1,j-1} \\
        \vdots \\
        w_{m,j-1}
    \end{bmatrix}
    + \sigma^2
    \begin{bmatrix}
        l(t_j) \\
        0 \\
        \vdots \\
        0 \\
        r(t_j)
    \end{bmatrix}.
\]
Ex. Apply the explicit Finite Difference Method to the wave equation with wave speed $c = 2$ and initial conditions $f(x) = \sin \pi x$ and $g(x) = l(x) = r(x) = 0$.

Figures show approximate solutions of the wave equation with $c = 2$. The explicit FDM is conditionally stable; step sizes have to be chosen carefully to avoid instability of the solver. Part (a) of the figure shows a stable choice of $h = 0.05$ and $k = 0.025$, while part (b) shows the unstable choice $h = 0.05$ and $k = 0.032$. The explicit FDM applied to the wave equation is unstable when the time step $k$ is too large relative to the space step $h$. 
The CFL Condition

The matrix form allows us to analyze the stability characteristics of the explicit FDM applied to the wave equation. The result of the analysis is stated in the following theorem.

**Theorem:**
The Finite Difference Method applied to the wave equation with wave speed $c > 0$ is stable if $\sigma = ck/h \leq 1$.

**Proof.** The equation (****) in vector form is

$$w_{j+1} = Aw_j - w_{j-1} + \sigma^2 s_j,$$

where $s_j$ holds the side conditions. Since $w_{j+1}$ depends on both $w_j$ and $w_{j-1}$, to analyze error, we rewrite the above scheme as

$$\begin{bmatrix} w_{j+1} \\ w_j \end{bmatrix} = \begin{bmatrix} A & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} w_j \\ w_{j-1} \end{bmatrix} + \sigma^2 \begin{bmatrix} s_j \\ 0 \end{bmatrix},$$
Error can not be raised as long as the eigenvalues of

\[
A' = \begin{bmatrix} A & -I \\ I & 0 \end{bmatrix}
\]

are bounded by 1 in absolute value. Let \( \lambda \) be different than 0, \((y, z)^T\) be an eigenvalue/eigenvector pair of \(A'\) such that

\[
\lambda y = Ay - z \\
\lambda z = y,
\]

which implies that

\[
Ay = \left( \frac{1}{\lambda} + \lambda \right) y,
\]

so that \( \mu = 1/\lambda + \lambda \) is an eigenvalue of \(A\). The eigenvalues of \(A\) lie between \(2 - 4\sigma^2\) (verify it as an exercise). The assumption that \(\sigma \leq 1\) implies that \(-2 \leq \mu \leq 2\). To finish, it need only be shown that, for a complex number \(\lambda\), the fact that \(1/\lambda + \lambda\) is real and has magnitude at most 2 implies that \(|\lambda| = 1\) (exercise). Thus the proof is done.
The quantity \( ck/h \) is called the **CFL number** of the method, after R. Courant, K. Friedrichs, and H. Lewy [1928]. In general, the CFL number must be at most 1 in order for the PDE solver to be stable. Since \( c \) is the wave speed, this means that the distance \( ck \) traveled by the solution in one time step should not exceed the space step \( h \). The figures (a) and (b) in the previous example illustrate CFL numbers of 1 and 1.28, respectively. The constraint \( ck \leq h \) is called the **CFL condition** for the wave equation. The related Theorem states that for the wave equation, the CFL condition implies stability of the Finite Difference Method. For more general hyperbolic equations, the CFL condition is necessary, but **not always sufficient** for stability. See Morton and Mayers (1996) for further details.